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Giuseppe PIPOLI

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

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INVERSE MEAN CURVATURE FLOW IN COMPLEX HYPERBOLIC SPACE

BY GIUSEPPE PIPOLI

ABSTRACT. – We consider the evolution by inverse mean curvature flow of a closed, mean convex and star-shaped hypersurface in the complex hyperbolic space. We prove that the flow is defined for any positive time, the evolving hypersurface stays star-shaped and mean convex. Moreover the induced metric converges, after rescaling, to a conformal multiple of the standard sub-Riemannian metric on the sphere. Finally we show that there exists a family of examples such that the Webster curvature of this sub-Riemannian limit is not constant.

RÉSUMÉ. – Nous considérons l'évolution par l'inverse de la courbure moyenne d'une surface étoilée, fermée et à courbure moyenne positive dans l'espace hyperbolique complexe. Nous montrons que le flot est défini pour tout temps positif et que la surface reste étoilée et à courbure moyenne positive. De plus, la métrique induite, après un changement d'échelle, converge vers un multiple conforme de la métrique sous-riemannienne standard sur la sphère de dimension impaire. Nous allons montrer l'existence d'exemples de données initiales telles que cette limite sous-riemannienne n'a pas courbure de Webster constante.

1. Introduction

During last years geometric flows of submanifolds of Riemannian manifolds have been studied intensively. In the class of expanding flows, the leading example is the inverse mean curvature flow. In this paper we consider the evolution by inverse mean curvature flow of real hypersurfaces of the complex hyperbolic space $\mathbb{CH}^n$, with $n \geq 2$. For any given smooth hypersurface $F_0 : \mathcal{M} \to \mathbb{CH}^n$, the solution of the inverse mean curvature flow with initial datum F_0 is a one-parameter family of smooth immersion $F : \mathcal{M} \times [0, T) \to \mathbb{CH}^n$ such that

$$(1.1)\qquad \begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{1}{H}\nu, \\ F(\cdot, 0) = F_0, \end{cases}$$

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where H is the mean curvature of $F_t = F(\cdot, t)$ and ν is the outward unit normal vector of $\mathcal{M}_t = F_t(\mathcal{M})$. It is the main tool used in the celebrated paper of Huisken and Ilmanen in [8] for proving the Penrose inequality.

In this paper we restrict to the class of star-shaped hypersurface in $\mathbb{CH}^n$. The inverse mean curvature flow of star-shaped hypersurfaces has been already studied in different ambient manifolds: for example the Euclidean space [4, 18], the hyperbolic space [6, 9], asymptotic hyperbolic spaces [13], rotationally symmetric spaces [2] and warped products [17, 19]. In any case it was proved that the flow is defined for any positive time and the evolving hypersurface stays star-shaped for all the life of the flow. Inverse mean curvature flow of star-shaped hypersurfaces in rank one symmetric spaces have been considered for different purposes in [11] too.

The geometry of the ambient manifold influences the nature of the limit of the induced metric. In fact Gerhardt [4] and Urbas [18] proved independently that, for any star-shaped hypersurface of the Euclidean space, the limit metric is, up to rescaling, always the standard round metric on the sphere. In [9] P.K. Hung and M.T. Wang showed that, when the ambient manifold is the hyperbolic space, the limit metric is not always round. More precisely it is a conformal multiple of the standard round metric on the sphere and so it is round only in special cases. The case studied in the present paper has some similarities with the previous results, but a new phenomenon appears: even after rescaling, the evolving metric blows up along a direction. Hence the limit metric is no more Riemannian, but only sub-Riemannian: it is defined only on a codimension-one distribution. The main theorem proved is the following.

Theorem 1.1. – *For any $\mathcal{M}_0$ closed, mean convex and $\mathbb{S}^1$-invariant star-shaped hypersurface in $\mathbb{CH}^n$ let $\mathcal{M}_t$ be its evolution by inverse mean curvature flow, let g_t be the induced metric on $\mathcal{M}_t$ and θ_t the induced contact form. Then:*

(1) *$\mathcal{M}_t$ is $\mathbb{S}^1$-invariant, star-shaped and mean convex for any time t;*

(2) *the flow is defined for any positive time;*

(3) *there is a smooth $\mathbb{S}^1$-invariant function f such that the rescaled induced metric $\tilde{g}_t = |\mathcal{M}_t|^{-\frac{1}{n}} g_t$ converges to a sub-Riemannian metric $\tilde{g}_\infty = e^{2f}\sigma_{sR}$ (i.e., a conformal multiple of the standard sub-Riemannian metric on the sphere $\mathbb{S}^{2n-1}$) and the rescaled contact form $\tilde{\theta} = |\mathcal{M}_t|^{-\frac{1}{n}} \theta_t$ converges to $\tilde{\theta}_\infty = e^{2f}\hat{\theta}$, where $\hat{\theta}$ is the standard contact form on the odd dimensional sphere;*

(4) *moreover there are examples of $\mathcal{M}_0$ such that the limit does not have constant Webster scalar curvature.*

After proving that the flow can be extended for any positive time, we have that the volume of $\mathcal{M}_t$ becomes arbitrary large and then $\mathcal{M}_t$ "explores" the structure at infinity of the ambient manifold as t tends to infinity. Hence the convergence to a sub-Riemannian metric is not surprising because the boundary at infinity of $\mathbb{CH}^n$ is $(\mathbb{S}^{2n-1}, \sigma_{sR})$, the one point compactification of the Heisenberg group of dimension $2n-1$ endowed with its standard sub-Riemannian metric. Different initial data explore this structure at infinity in different ways, but our result shows that we remain in the conformal class of σ_{sR}.

Obviously for any finite time t, $\tilde{g}_t$ is a Riemannian metric, but there is a direction in which the metric is blowing up. This special direction is $J\nu$, where J is the complex structure of $\mathbb{CH}^n$ and ν is the outward unit normal vector field of $\mathcal{M}_t$. Note that, since we are considering only submanifolds of codimension one, $J\nu$ is for sure tangent to $\mathcal{M}_t$. One of the main difficulties in generalizing the previous results is to describe the contribution of this special direction. Its presence gives also a new phenomenon not present in the previous literature. The second fundamental form converges to that of an horosphere with an exponential speed but, unlike for example [4, 17, 19], we have that the speed is not the same for any initial datum: very symmetric hypersurfaces converge twice as fast as the generic $\mathbb{S}^1$-invariant hypersurface (see Remark 6.6 below for more details).

If we try to study the limit of the sectional curvature of this family of metric $\tilde{g}_t$, it always diverges: this is a general behavior when we try to approximate a sub-Riemannian metric with a family of Riemannian metrics. For that reason another notion of curvature is required. We will use in particular the Webster curvature.

It is very easy to find hypersurfaces of $\mathbb{CH}^n$ such that $\tilde{g}_\infty$ has constant Webster curvature. It is the case of the geodesic spheres: as we will see in detail in Section 4, the evolution of a geodesic sphere is a family of geodesic spheres and the function f is constant. On the other hand, the search for an example with a non-trivial limit is much more challenging. The main tool for studying the roundness of the limit is the following Brown-York like quantity: for any star-shaped hypersurface $\mathcal{M}$

$$Q(\mathcal{M}) = |\mathcal{M}|^{-1+\frac{1}{n}} \int_{\mathcal{M}} \left(H - \hat{H}\right) d\mu,$$

where $|\mathcal{M}|$ is the volume of $\mathcal{M}$ and, if ρ is the radial function defining $\mathcal{M}$, $\hat{H}$ is the value of the mean curvature of a geodesic sphere of radius ρ (see (3.8) for the explicit definition). Q gives a measure of how $\mathcal{M}$ is far to being a geodesic sphere. It is not a measure in the strict sense because Q has not a sign and, even if it is zero for geodesic spheres, it is not in general truly the opposite. In the final section of this paper we found the desired non-trivial examples estimating the behavior of Q along the inverse mean curvature flow.

This paper is organized as follows. In Section 2 we collect some preliminaries and we fix some notations. In Section 3 we compute the main geometric quantities for a star-shaped hypersurface in $\mathbb{CH}^n$, like the induced metric, the second fundamental form and the mean curvature. In Section 4 we have a simple but meaningful example, i.e., the evolution of the geodesic spheres. In Section 5 we estimate the norm of the gradient of the radial function. As consequence we have that the property of being star-shaped and the mean convexity are preserved by the flow. The study of the derivatives of the radial function continues in Sections 6 and 7. In particular we prove that the solution of the flow is defined for any positive time. Section 8 is devoted to the proof of the convergence of the rescaled induced metric to a sub-Riemannian limit. In the last section we conclude the proof of Theorem 1.1 by studying the Webster curvature of the limit metric and giving a family of non-trivial examples.

Finally we would like to announce that the ideas developed in the present paper have been extended in [16] in the case of the next rank one symmetric space, that is the quaternionic hyperbolic space. An analogous of Theorem 1.1 holds in this other setting too.