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ON SUBLINEAR BILIPSCHITZ EQUIVALENCE OF GROUPS

BY YVES DE CORNULIER

ABSTRACT. – We discuss the notion of sublinearly bilipschitz equivalences (SBE), which generalize quasi-isometries, allowing some additional terms that behave sublinearly with respect to the distance from the origin. Such maps were originally motivated by the fact they induce bilipschitz homeomorphisms between asymptotic cones. We prove here that for hyperbolic groups, they also induce Hölder homeomorphisms between the boundaries. This yields many basic examples of hyperbolic groups that are pairwise non-SBE. Besides, we check that subexponential growth is an SBE-invariant.

The central part of the paper addresses nilpotent groups. While classification up to sublinearly bilipschitz equivalence is known in this case as a consequence of Pansu's theorems, its quantitative version is not. We introduce a computable algebraic invariant $e = e_G < 1$ for every such group G, and check that G is $O(r^e)$-bilipschitz equivalent to its associated Carnot group. Here $r \mapsto r^e$ is a quantitive sublinear bound.

Finally, we define the notion of large-scale contractable and large-scale homothetic metric spaces. We check that these notions imply polynomial growth under general hypotheses, and formulate conjectures about groups with these properties.

RÉSUMÉ. – On étudie les équivalences sous-linéairement bilipschitziennes (SBE), qui généralisent les quasi-isométries, en autorisant un terme d'erreur sous-linéaire par rapport à la distance à l'origine. L'introduction de telles applications a été initialement motivée par le fait qu'elles induisent des homéomorphismes bilipschitziens au niveau des cônes asymptotiques. On démontre ici que pour les groupes hyperboliques, elles induisent également des homéomorphismes hölderiens entre leurs bords de Gromov. Ceci permet d'obtenir de nombreux exemples de groupes hyperboliques qui ne sont pas SBE entre eux. En outre, on vérifie qu'être à croissance sous-exponentielle est invariant par SBE.

La partie centrale de l'article concerne les groupes nilpotents. Leur classification à SBE près se déduit des travaux de Pansu des années 80, mais la version quantitative reste à étudier. On introduit un invariant algébrique calculable $e = e_G < 1$ pour les groupes nilpotents G et on vérifie que G est toujours $O(r^e)$-SBE à son groupe Carnot-gradué associé: la fonction $r \mapsto r^e$ est une borne sous-linéaire quantitative.

Enfin, on introduit les notions d'espaces métriques contractables à grande échelle, et homothétique à grande échelle. On vérifie, sous des hypothèses très générales, qu'elles impliquent être à croissance polynomiale, et on formule des conjectures sur les groupes ayant ces propriétés.

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1. Introduction

1.A. Sublinearly Lipschitz maps

We consider here some functions between metric spaces, generalizing large-scale Lipschitz maps and quasi-isometries.

Let v be a real-valued function on $\mathbf{R}_+$. For our purposes, typical examples of v are $v(r) = r^\alpha$ for some $\alpha \in [0,1]$, or $v(r) = \log(r)$. See § 2 for precise (mild) hypotheses. We assume here $v \geq 1$, up to replacing v with $\max(v,1)$ if necessary.

We say that a map $f : X \to Y$ between metric spaces is $O(v)$-Lipschitz if it satisfies

$$d(f(x), f(x')) \leq Cd(x,x') + C'v(|x| + |x'|), \qquad \forall x, x' \in X,$$

for some constants $C, C' > 0$. Here, $|x|$ denotes the distance from x to some base-point of X (fixed once and for all, but whose choice does not matter). We also say that f is $o(v)$-Lipschitz if it is $O(v')$-Lipschitz for some $v' = o(v)$; in particular, for $v(r) = r$, we call $o(r)$-Lipschitz maps sublinearly Lipschitz maps; they were introduced in [10] under the name "cone-Lipschitz maps".

For instance, $O(1)$-Lipschitz maps are known as large-scale Lipschitz maps, and occur naturally in the large-scale category, whose isomorphisms are quasi-isometries; see for instance [13, Chap. 3].

We are especially interested in $O(v)$-Lipschitz maps when $v(r) = o(r)$ (that is, sublinearly Lipschitz maps). Indeed, for instance $O(r)$-Lipschitz maps are just maps with a radial control $|f(x)| \leq C|x| + C'$ and are thus of limited interest.

There is a natural equivalence relation on the set of $O(v)$-Lipschitz maps $X \to Y$, called $O(v)$-closeness. Namely, f, f' are $O(v)$-close if $d(f(x), f'(x)) \leq C''v(|x|)$ for all $x \in X$, and some $C'' > 0$. Similarly, f, f' are $o(v)$-close if they are $O(v')$-close for some $v' = o(v)$.

The sublinearly Lipschitz category consists in metric spaces as objects, with the set of arrows $X \to Y$ being the set of $o(r)$-Lipschitz maps up to $o(r)$-equivalence. This category was introduced in [10] with the following motivation: taking asymptotic cones (with respect to a given scaling sequence and ultrafilter) yields a functor from this category to the category of metric spaces with Lipschitz maps. Moreover, this is, in a precise sense [10, Prop. 2.9], the largest setting for which such functors can be defined. Isomorphisms in this category are called sublinearly Lipschitz equivalences, or SBE maps ("cone-bilipschitz equivalences" in [10]). A simple verification shows that a map $f : X \to Y$ is an SBE if and only if there exists a (locally bounded above) function $v = o(r)$ such that f satisfies the following three conditions:

— f is $O(v)$-Lipschitz: there exist constants $c, C > 0$ such that for all $x, x' \in X$ one has
$$d(f(x), f(x')) \leq cd(x,x') + Cv(|x| + |x'|);$$

— f is $O(v)$-expansive: there exist constants $c', C' > 0$ such that for all $x, x' \in X$ one has
$$c'd(x,x') - C'v(|x| + |x'|) \leq d(f(x), f(x'));$$

— f is $O(v)$-surjective: there exists a constant $C'' > 0$ such that for all $y \in Y$, one has
$$d(y, f(X)) \leq C''v(|y|).$$

Note that multiplying v by a scalar (depending on f) allows to get rid of the constants C, C', C''.

EXAMPLE 1.1. – 1) If π is any sublinear function $\mathbf{R} \to \mathbf{R}$, then $x \mapsto x + \pi(x)$ is an SBE $\mathbf{R} \to \mathbf{R}$ (being $o(r)$-close to the identity map).

2) Let X be the set of square integers in $\mathbf{R}_+$ and let f be the map $x \mapsto \lfloor\sqrt{x}\rfloor^2$ from $\mathbf{R}_+$ to X. Then f is an SBE, but is not $o(r)$-close to any large-scale Lipschitz map: indeed by a simple geodesic argument, every large-scale Lipschitz map $\mathbf{R}_+ \to X$ is bounded and in particular X and $\mathbf{R}_+$ are not quasi-isometric.

There are more elaborate examples, which are important in geometric group theory. For simply connected nilpotent Lie groups, the classification up to quasi-isometry is conjectural: it is expected that quasi-isometric implies isomorphic. Nevertheless, Pansu obtained the first non-trivial quasi-isometric rigidity results by proving important theorems about their asymptotic cones in the eighties, which can be restated in terms of SBEs. Namely every simply connected nilpotent Lie group is SBE to a Carnot group (called its associated Carnot-graded group) [27], and any two SBE Carnot groups are isomorphic [28]. Later, Shalom [30] proved, with unrelated methods, that the Betti numbers of the Lie algebra of a simply connected nilpotent Lie group are quasi-isometry invariants, while taking associated Carnot-graded group does not always preserves the Betti numbers. Thus there exist simply connected nilpotent Lie groups (with lattices) that are SBE but not quasi-isometric. SBEs allow to restate Pansu's theorems with no reference to asymptotic cones (the asymptotic cone theorems, also related to Goodman's earlier work [18], being corollaries), but also yields other interpretations. We come back to this topic in § 1.F.

One can naturally generalize the sublinearly Lipschitz category and define, in a similar way, the $O(v)$-Lipschitz category and the $o(v)$-Lipschitz category. In particular, the $O(1)$-Lipschitz category is known as the large-scale Lipschitz category. Thus these interpolate between the large-scale category and the sublinearly Lipschitz category. There are obvious inclusion functors from the $O(v)$-category to the $O(v')$-category whenever $v = O(v')$ (they are usually not faithful, because of the equivalence relation). Isomorphisms in the $O(v)$-Lipschitz or $o(v)$-Lipschitz category are called $O(v)$-SBE or $o(v)$-SBE (assuming that $v = o(r)$).

For instance, it is established in [10] that every connected Lie group is $O(\log r)$-SBE to a Lie group of the form $G = N \rtimes E$ with both N, E simply connected nilpotent Lie groups, N being exponentially distorted in G and E acting in a diagonalizable way on the Lie algebra of N.

1.B. SBEs and growth

In § 3, we prove the following:

THEOREM 1.2. – 1) *Subexponential growth is SBE-invariant among connected graphs of bounded valency, and in particular among compactly generated locally compact groups.*

2) *(Folklore up to the formulation) Polynomial growth is SBE-invariant among compactly generated locally compact groups.*