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Hubert LACOIN

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

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PINNING AND DISORDER RELEVANCE FOR THE LATTICE GAUSSIAN FREE FIELD II: THE TWO DIMENSIONAL CASE

BY HUBERT LACOIN

ABSTRACT. – This paper continues a study initiated in [35], on the localization transition of a lattice free field on $\mathbb{Z}^d$ interacting with a quenched disordered substrate that acts on the interface when its height is close to zero. The substrate has the tendency to localize or repel the interface at different sites. A transition takes place when the average pinning potential h goes past a threshold h_c: this critical value separates a delocalized phase $h < h_c$, where the field is macroscopically repelled by the substrate from a localized one $h > h_c$ where the field sticks to the substrate. Our goal is to investigate the effect of the presence of disorder on this phase transition. We focus on the two dimensional case ($d = 2$) for which we had obtained so far only limited results. We prove that the value of $h_c(\beta)$ is the same as for the annealed model, for all values of the disorder intensity β. Moreover we prove that, in contrast with the case $d \geq 3$ where the free energy has a quadratic behavior near the critical point, the phase transition is of infinite order

$$\lim_{u \to 0+} \frac{\log \mathrm{F}(\beta, h_c(\beta) + u)}{(\log u)} = \infty.$$

An analogous result is presented for the two dimensional co-membrane model.

RÉSUMÉ. – Cet article approfondit l'étude (commencée dans [35]) de la transition de localisation pour un champ libre gaussien défini sur le réseau $\mathbb{Z}^d$ en interaction avec un substrat désordonné qui affecte les points situés proches de la hauteur zéro. Le substrat peut avoir un effet attracteur ou répulsif selon le site considéré. Une transition a lieu lorsque le potentiel moyen d'interaction h dépasse un certain seuil h_c: cette valeur critique définit une phase délocalisée $h < h_c$, au sein de laquelle le champ est globalement repoussé par le substrat, et une phase localisée $h > h_c$ ou le champ adhère au substrat. Notre objectif est d'évaluer les effets de la présence de désordre pour cette transition de phase. Nous nous concentrons sur le cas bi-dimensionnel ($d = 2$), et démontrons que la valeur du point critique $h_c(\beta)$ coincide avec celle du modèle moyenné (ou *annealed*), et ce quelle que soit la valeur de l'intensité du désordre β. De plus, nous démontrons que, contrairement au cas $d \geq 3$ pour lequel l'énergie libre a un comportement quadratique au voisinage du point critique, la transition de phase est ici d'ordre infini

$$\lim_{u \to 0+} \frac{\log \mathrm{F}(\beta, h_c(\beta) + u)}{(\log u)} = \infty.$$

Un résultat analogue est exposé pour le modèle de co-membrane bi-dimensionnelle.

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1. Introduction

The aim of statistical mechanics is to obtain a qualitative understanding of natural phenomena of phase transitions by the study of simplified models, often built on a lattice. In general the Hamiltonian of a model of statistical mechanics is left invariant by the lattice symmetries: a prototypical example being the Ising model describing a ferromagnet.

However, one might argue that materials which are found in nature are usually not completely homogeneous and for this reason, physicists were led to considering systems in which the interaction terms, for example the potentials between nearest neighbor spins, are chosen by sampling a random field—which we call *disorder*—with good ergodic properties, often a field of independent identically distributed random variables. An important question which arises is thus whether the results concerning the phase transition obtained for a model with homogeneous interactions referred to as *the pure system* (e.g., Onsager's solution of the two dimensional Ising Model [47]) remain valid when a system where randomness of a very small amplitude is introduced.

In [39] A. B. Harris gave a strikingly simple heuristical argument, based on renormalization theory consideration, to predict the effect of the introduction of a small amount of the system: in substance Harris' criterion predicts that if the phase transition of the pure system is sufficiently smooth, it will not be affected by small perturbation (disorder is then said to be *irrelevant*), while in the other cases the behavior of the system is affected by an arbitrary small addition of randomness (disorder is *relevant*). To be complete, let us mention also the existence of a boundary case for which the criterion yields no prediction (the *marginal disorder* case). The criterion however does not give a precise prediction concerning the nature of the phase transition when the disorder is relevant.

The mathematical verification of the Harris criterion is a very challenging task in general. In the first place, it can only be considered for the few special models of statistical mechanics for which we have a rigorous understanding of the critical properties of the pure system. In the last twenty years this question has been addressed, first by theoretical physicists (see e.g., [27] and references therein) and then by mathematicians [4, 5, 3, 7, 26, 37, 36, 38, 42, 48] (see also [32, 33] for reviews) for a simple model of a 1-dimensional interface interacting with a substrate: for this model the interface is given by the graph of a random walk which takes random energy rewards when it touches a defect line. In this case, the pure system has the remarkable quality of being what physicists call *exactly solvable*, meaning that there exists an explicit expression for the free energy [29].

This model under consideration in the present paper can be seen as a high dimension generalization of the random walk pinning model. The random walk is replaced by a random field $\mathbb{Z}^d \to \mathbb{R}$, and the random energies are collected when the graph of the field is close to the hyper-plane $\mathbb{Z}^d \times \{0\}$. While the pure model is not exactly solvable in that case, it has been studied in details and the nature of the phase transition is well known [13, 15, 17, 19, 50].

On the other hand, the study of the disordered version of the model is much more recent [22, 23, 35, 34]. In [35], we gave a close to complete description of the free energy diagram of the disordered model when $d \geq 3$:

- We identified the value of the disordered critical point, which is shown to coincide with that of the associated annealed model, regardless of the amplitude of disorder.

— We proved that for Gaussian disorder, the behavior of the free energy close to h_c is quadratic, in contrast with the annealed model for which the transitition is of first order.

— In case of general disorder, we proved that the quadratic upper bound still holds, and found a polynomial lower bound with a different exponent.

Let us stress that the heuristic of our proof strongly suggests that the behavior of the free energy should be quadratic for a suitable large class of environments (those who satisfy a second moment assumption similar to (2.5)).

In the present paper, we choose to attack the case $d = 2$, for which only limited results were obtained so far. We have seen in the proof of the main result [35] that the critical behavior of the model is very much related to the extremal process of the field. The quadratic behavior of the free energy in [35, Theorem 2.2] comes from the fact that high level sets of the Gaussian free field for $d \geq 3$ look like a uniformly random set with a fixed density (see [21]). In dimension 2 however, the behavior of the extremal process is much more intricate, with a phenomenon of clustering in the level sets (see [11, 28, 24] or also [6] for a similar phenomenon for branching Brownian Motion). This yields results of a very different nature.

2. Model and results

Given Λ a finite subset of $\mathbb{Z}^d$, we let $\partial\Lambda$ denote the internal boundary of Λ, $\mathring{\Lambda}$ the set of interior points of Λ, and $\partial^-\Lambda$ the set of points which are adjacent to the boundary,

$$\begin{aligned}\partial\Lambda &:= \{x \in \Lambda : \exists y \notin \Lambda,\ x \sim y\},\\ \mathring{\Lambda} &:= \Lambda \setminus \partial\Lambda,\\ \partial^-\Lambda &:= \{x \in \mathring{\Lambda} : \exists y \in \partial\Lambda,\ x \sim y\}.\end{aligned} \tag{2.1}$$

In general some of these sets could be empty, but throughout this work Λ is going to be a large square. Given $\widehat{\phi} : \mathbb{Z}^d \to \mathbb{R}$, we define $\mathbf{P}_\Lambda^{\widehat{\phi}}$ to be the law of the lattice Gaussian free field $\phi = (\phi_x)_{x\in\Lambda}$ with boundary condition $\widehat{\phi}$ on $\partial\Lambda$. The field ϕ is a random function from Λ to $\mathbb{R}$. It satisfies

$$\phi_x := \widehat{\phi}_x \quad \text{for every } x \in \partial\Lambda, \tag{2.2}$$

and the distribution of $(\phi_x)_{x\in\mathring{\Lambda}}$ is given by

$$\mathbf{P}_\Lambda^{\widehat{\phi}}(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}_\Lambda^{\widehat{\phi}}} \exp\left(-\frac{1}{2} \sum_{\substack{(x,y)\in(\Lambda)^2\setminus(\partial\Lambda)^2\\ x\sim y}} \frac{(\phi_x - \phi_y)^2}{2}\right) \prod_{x\in\mathring{\Lambda}} \mathrm{d}\phi_x, \tag{2.3}$$

where $\prod_{x\in\mathring{\Lambda}} \mathrm{d}\phi_x$ denotes the Lebesgue measure on $\mathbb{R}^{\mathring{\Lambda}}$ and

$$\mathcal{Z}_\Lambda^{\widehat{\phi}} := \int_{\mathbb{R}^{\mathring{\Lambda}}} \exp\left(-\frac{1}{2} \sum_{\substack{(x,y)\in(\Lambda)^2\setminus(\partial\Lambda)^2\\ x\sim y}} \frac{(\phi_x - \phi_y)^2}{2}\right) \prod_{x\in\mathring{\Lambda}} \mathrm{d}\phi_x \tag{2.4}$$