389 ASTÉRISQUE

2017

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque est un périodique de la Société mathématique de France

Numéro 389

Diffusion

Maison de la SMF
B.P. 67
13274 Marseille CEDEX 9
France
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AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs 2017

Vente au numéro : 35 € ($ 52)
Abonnement électronique : 500 € ($ 750)
Abonnement avec supplément papier : 657 €, *hors Europe :* 699 € ($ 1049)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

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astsmf@ihp.fr • http://smf.emath.fr/

ISSN : print 0303-1179, electronic 2492-5926

ISBN 978-2-85629-854-1

Directeur de la publication : Stéphane SEURET

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2010 ***Mathematics Subject Classification.*** **—** 35B15, 47B35, 37K15.

Key words and phrases. **—** Cubic Szegő equation, integrable system, nonlinear Schrödinger equation, Hankel operator, spectral analysis.

The authors are grateful to Z. Hani, T. Kappeler, S. Kuksin, V. Peller, A. Pushnitski and N. Tzvetkov for valuable discussions and comments about earlier versions of this work, to L. Baratchart for pointing them references [**1**] and [**3**], and to A. Nazarov for references concerning Bateman–type formulae. They also thank the anonymous referees for making helpful suggestions and comments. The first author is supported by ANR ANAE 13-BS01-0010-03.

THE CUBIC SZEGŐ EQUATION AND HANKEL OPERATORS

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***Abstract.* —** This monograph is devoted to the dynamics on Sobolev spaces of the cubic Szegő equation on the circle $\mathbb{S}^1$,

$$i\partial_t u = \Pi(|u|^2 u).$$

Here Π denotes the orthogonal projector from $L^2(\mathbb{S}^1)$ onto the subspace $L^2_+(\mathbb{S}^1)$ of functions with nonnegative Fourier modes. We construct a nonlinear Fourier transformation on $H^{1/2}(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$ allowing to describe explicitly the solutions of this equation with data in $H^{1/2}(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$. This explicit description implies almost-periodicity of every solution in this space. Furthermore, it allows to display the following turbulence phenomenon. For a dense G_δ subset of initial data in $C^\infty(\mathbb{S}^1)\cap L^2_+(\mathbb{S}^1)$, the solutions tend to infinity in H^s for every $s > \frac{1}{2}$ with super–polynomial growth on some sequence of times, while they go back to their initial data on another sequence of times tending to infinity. This transformation is defined by solving a general inverse spectral problem involving singular values of a Hilbert–Schmidt Hankel operator and of its shifted Hankel operator.

***Résumé* (Équation de Szegő cubique et opérateurs de Hankel)**

Cette monographie est consacrée à l'étude de la dynamique, dans les espaces de Sobolev, de l'équation de Szegő cubique sur le cercle $\mathbb{S}^1$,

$$i\partial_t u = \Pi(|u|^2 u),$$

où Π désigne le projecteur orthogonal de $L^2(\mathbb{S}^1)$ sur le sous-espace $L^2_+(\mathbb{S}^1)$ des fonctions à modes de Fourier positifs ou nuls. On construit une transformée de Fourier non linéaire sur $H^{1/2}(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$ permettant de résoudre explicitement cette équation avec données initiales dans $H^{1/2}(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$. Ces formules explicites entraînent la presque périodicité des solutions dans cet espace. Par ailleurs, elles permettent de mettre en évidence le phénomène de turbulence suivant. Pour un G_δ dense de données initiales de $C^\infty(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$, les solutions tendent vers l'infini à vitesse sur–polynomiale en norme $H^s(\mathbb{S}^1)$ pour tout $s > \frac{1}{2}$ sur une suite de temps, alors qu'elles retournent vers leur donnée initiale sur une autre suite de temps tendant vers l'infini. Cette transformation est définie *via* la résolution d'un problème spectral inverse lié aux valeurs singulières d'un opérateur de Hankel Hilbert-Schmidt et de son opérateur décalé.

CONTENTS

CHAPTER 1

INTRODUCTION

The large time behavior of solutions to Hamiltonian partial differential equations is an important problem in mathematical physics. In the case of finite dimensional Hamiltonian systems, many features of the large time behavior of trajectories are described using the topology of the phase space. For a given infinite dimensional system, several natural phase spaces, with different topologies, can be chosen, and the large time properties may strongly depend on the choice of such topologies. For instance, it is known that the cubic defocusing nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = |u|^2 u$$

posed on a Riemannian manifold M of dimension $d = 1, 2, 3$ with sufficiently uniform properties at infinity, defines a global flow on the Sobolev spaces $H^s(M)$ for every $s \geq 1$. In this case, a typical large time behavior of interest is the boundedness of trajectories. On the energy space $H^1(M)$, the conservation of energy trivially implies that all the trajectories are bounded. On the other hand, the existence of unbounded trajectories in $H^s(M)$ for $s > 1$ as well as bounds of the growth in time of the $H^s(M)$ norms is a long standing problem [**7**]. As a way to detect and to measure the transfer of energy to small scales, this problem is naturally connected to wave turbulence. In [**6**] and [**38**], it has been established that big H^s norms grow at most polynomially in time. Existence of unbounded trajectories only recently [**22**] received a positive answer in some very special cases, while several instability results were already obtained [**9**], [**20**], [**21**], [**24**], [**19**]. Natural model problems for studying these phenomena seem to be those for which the calculation of solutions is the most explicit, namely integrable systems. Continuing with the example of nonlinear Schrödinger equations, a typical example is the one dimensional cubic nonlinear Schrödinger defocusing equation ([**43**]). However, in this case, the set of conservation laws is known to control the whole regularity of the solution, so that all the trajectories of $H^s(M)$ are bounded in $H^s(M)$ for every nonnegative integer s. In fact, when the equation is posed on the circle, the recent results of [**18**] show that, for every such s, the trajectories in $H^s(M)$ are almost periodic in $H^s(M)$.

The goal of this monograph is to study an integrable infinite dimensional system, connected to a nonlinear wave equation, with a dramatically different large time behavior of its trajectories according to the regularity of the phase space.

Following [**30**] and [**42**], it is natural to change the dispersion relation by considering the family of equations, $\alpha < 2$,

$$i\partial_t u - |D|^\alpha u = |u|^2 u$$

posed on the circle $\mathbb{S}^1$, where the operator $|D|^\alpha$ is defined by

$$\widehat{|D|^\alpha u}(k) = |k|^\alpha \hat{u}(k)$$

for every distribution $u \in \mathcal{D}'(\mathbb{S}^1)$. Numerical simulations in [**30**] and [**42**] suggest weak turbulence. For $1 < \alpha < 2$, it is possible to recover at most polynomial growth of the Sobolev norms ([**39**]). In the case $\alpha = 1$, the so-called half-wave equation

$$i\partial_t u - |D| u = |u|^2 u \tag{1.0.1}$$

defines a global flow on $H^s(\mathbb{S}^1)$ for every $s \geq \frac{1}{2}$ ([**11**] — see also [**36**]). In that case, the only available bound of the H^s-norms is e^{ct^2} ([**39**]), due to the lack of dispersion. Therefore, the special case $\alpha = 1$ seems to be a more favorable framework for displaying wave turbulence effects.

Notice that this equation can be reformulated as a system, using the Szegõ projector Π defined on $\mathcal{D}'(\mathbb{S}^1)$ as

$$\widehat{\Pi u}(k) = \mathbf{1}_{k \geq 0} \hat{u}(k).$$

Indeed, setting $u_+ := \Pi u$, $u_- := (I - \Pi)u$, equation (1.0.1) is equivalent to the system

$$\begin{cases} i(\partial_t + \partial_x)u_+ & = \Pi(|u|^2 u) \\ i(\partial_t - \partial_x)u_- & = (I - \Pi)(|u|^2 u) \end{cases}$$

Furthermore, if the initial datum u_0 satisfies $u_0 = \Pi u_0$ and belongs to $H^s(\mathbb{S}^1)$, $s > 1$, with a small norm ε, then the corresponding solution u is approximated by the solution v of

$$i(\partial_t + \partial_x)v = \Pi(|v|^2 v), \tag{1.0.2}$$

for $|t| \leq \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}$ [**11**] [**36**]. In other words, the first nonlinear effects in the above system arise through a decoupling of the two equations. Notice that an elementary change of variable in equation (1.0.2) reduces it to

$$i\partial_t u = \Pi(|u|^2 u). \tag{1.0.3}$$

Equation (1.0.3) therefore appears as a model evolution for describing the dynamics of the half-wave equation (1.0.1). We introduced it in [**10**] under the name *cubic*