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THE LANGLANDS-WEISSMAN PROGRAM FOR BRYLINSKI-DELIGNE EXTENSIONS

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Abstract. — We describe an evolving and conjectural extension of the Langlands program for a class of nonlinear covering groups of algebraic origin studied by Brylinski and Deligne. In particular, we describe the construction of an L-group extension of such a covering group (over a split reductive group) due to Weissman, study some of its properties and discuss a variant of it. Using this L-group extension, we describe a local Langlands correspondence for covering (split) tori and unramified genuine representations, using work of Savin, McNamara, Weissman and W.-W. Li. We then define the notion of automorphic (partial) L-functions attached to genuine automorphic representations of the covering groups of Brylinski and Deligne. Finally, we see how the L-group formalism explains certain anomalies in the representation theory of covering groups and examine some examples of Langlands functoriality such as base change.

***Résumé* (Le programme de Langlands-Weissman pour les extensions de Brylinski-Deligne)**

Nous décrivons une extension conjecturale, en evolution, du programme de Langlands pour une classe de revêtements de groupes réductifs d'origine algébrique, étudiés par Brylinski et Deligne. Nous décrivons, en particulier, la construction, due à Weissman, d'une extension de L-groupe d'un tel revêtement de groupe (au-dessus d'un groupe réductif déployé). Nous étudions certaines de ses propriétés et discutons d'une variante de celui-ci. En utilisant cette extension de L-groupe, à l'aide du travail de Savin, McNamara, Weissman et W.-W. Li, nous décrivons une correspondance de Langlands locale pour des revêtements des tores (déployés) et pour les représentations non-ramifiées spécifiques. Nous définissons ensuite la notion de L-fonctions automorphes (partielles) attachées aux représentations automorphes spécifiques pour les groupes de Brylinski et Deligne. Enfin, nous verrons comment le formalisme de L-groupe explique certaines anomalies dans la théorie des représentations des revêtements de groupes réductifs et examinons quelques exemples de fonctorialité de Langlands tels que le changement de base.

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1. Introduction

One of the goals of the local Langlands program is to provide an arithmetic classification of the set of isomorphism classes of irreducible representations of a locally compact group $G = \mathbb{G}(F)$, where $\mathbb{G}$ is a connected reductive group over a local field F. Analogously, if k is a number field with ring of adeles $\mathbb{A}$, the global Langlands program postulates a classification of automorphic representations of $\mathbb{G}(\mathbb{A})$ in terms of Galois representations. In this proposed arithmetic classification, which has been realized in several important instances, a key role is played by the L-group ${}^L\mathbb{G}$ of $\mathbb{G}$. This key notion was introduced by Langlands in his re-interpretation of the Satake isomorphism in the theory of spherical functions and used by him to introduce the notion of *automorphic L-functions*. One of the main goals of this paper is to do the same for a class of nonlinear covering groups of "algebraic origin" studied by Brylinski-Deligne [**15**].

1.1. Covering groups. — The theory of the L-group is so far confined to the case when $\mathbb{G}$ is a connected reductive linear algebraic group. On the other hand, since Steinberg's beautiful paper [**59**], the structure theory of nonlinear covering groups of G (i.e., topological central extensions of G by finite groups) have been investigated by many mathematicians, notably Moore [**48**], Matsumoto [**42**], Deodhar [**21**], Deligne [**20**], Prasad-Raghunathan [**50, 51, 52**], and its relation to the reciprocity laws of abelian class field theory has been noted. In addition, nonlinear covering groups of G have repeatedly made their appearance in representation theory and the theory of automorphic forms. This goes way back to Jacobi's construction of his theta function, a holomorphic modular form of weight $1/2$, and a more recent instance is the work of Kubota [**30**] and the Shimura correspondence between integral and half integral weight modular forms. Both these examples concern automorphic forms and representations of the metaplectic group $\mathrm{Mp}_2(F)$, which is a nonlinear double cover of $\mathrm{SL}_2(F) = \mathrm{Sp}_2(F)$. As another example, the well-known Weil representation of $\mathrm{Mp}_{2n}(F)$ gives a representation theoretic incarnation of theta functions and has been a very useful tool in the construction of automorphic forms. Finally, much of Harish-Chandra's theory of local harmonic analysis and Langlands' theory of Eisenstein series continue to hold for such nonlinear covering groups (see [**47**] and [**36**]).

It is thus natural to wonder if the framework of the Langlands program can be extended to encompass the representation theory and the theory of automorphic forms of covering groups. There have been many attempts towards this end, such as Flicker [**23**], Kazhdan-Patterson [**28, 29**], Flicker-Kazhdan [**24**], Adams [**1, 2**], Savin [**57**] among others. However, these attempts have tended to focus on the treatment of specific families of examples rather than a general theory. This is understandable, for what is lacking is a structure theory which is sufficiently functorial. For example, the classification of nonlinear covering groups given in [**48, 21, 50, 51**] is given only when $\mathbb{G}$ is simply-connected and isotropic, in which case a universal cover exists.

1.2. Brylinski-Deligne theory. — A functorial structure theory was finally developed by Brylinski and Deligne [**15**]. More precisely, Brylinski-Deligne considered the category of multiplicative $\mathbb{K}_2$-torsors on a connected reductive group $\mathbb{G}$ over F; these are extensions of $\mathbb{G}$ by the sheaf $\mathbb{K}_2$ of Quillen's K_2 group in the category of sheaves of groups on the big Zariski site of $\mathrm{Spec}(F)$:

$$1 \longrightarrow \mathbb{K}_2 \longrightarrow \overline{\mathbb{G}} \longrightarrow \mathbb{G} \longrightarrow 1.$$

In other words, Brylinski and Deligne started with an extension problem in the world of algebraic geometry. Some highlights of [**15**] include:

- an elegant and functorial classification of this category in terms of enhanced root theoretic data, much like the classification of split connected reductive groups by their root data.
- the description of a functor from the category of multiplicative $\mathbb{K}_2$-torsors $\overline{\mathbb{G}}$ on $\mathbb{G}$ (together with an integer n such that $\#\mu_n(F) = n$, which determines the degree of the covering) to the category of topological central extensions $\overline{G}$ of G:

  $$1 \longrightarrow \mu_n \longrightarrow \overline{G} \longrightarrow G \longrightarrow 1.$$

  These topological central extensions may be considered of "algebraic origin" and can be constructed using cocycles which are essentially algebraic in nature.
- though this construction does not exhaust all topological central extensions, it captures a sufficiently large class of such extensions, and essentially all interesting examples which have been investigated so far; for example, it captures all such coverings of G when $\mathbb{G}$ is split and simply-connected (except perhaps in the case of type C_r over real numbers).

We shall give a more detailed discussion of the salient features of the Brylinski-Deligne theory in §2 and §3. Hence, the paper [**15**] provides a structure theory which is essentially algebraic and categorical, and may be perceived as a natural extension of Steinberg's original treatment [**59**] from the split simply connected case to general reductive groups.

1.3. Dual and L-groups. — One should expect that such a natural structure theory would elucidate the study of representations and automorphic forms of the Brylinski-Deligne covering groups $\overline{G}$, henceforth referred to as BD covering groups. Indeed, Brylinski and Deligne wrote in the introduction of [**15**]: "We hope that for k a global field, this will prove useful in the study of metaplectic automorphic forms, that is, the harmonic analysis of functions on $\tilde{G}(\mathbb{A})/G(k)$".

The first person to fully appreciate this is probably our colleague M. Weissman. In a series of papers [**66, 67, 27**], Weissman systematically exploited the Brylinski-Deligne theory to study the representation theory of covering tori, the unramified representations and the depth zero representations. This was followed by the work of several authors who discovered a "Langlands dual group" $\overline{G}^{\vee}$ for a BD covering group $\overline{G}$ (with $\mathbb{G}$ split) from different considerations. These include the work of Finkelberg-Lysenko [**22**] and Reich [**54**] in the framework of the geometric Langland program and the work of McNamara [44, 45] who established a Satake isomorphism and interpreted it in terms of the dual group $\overline{G}^{\vee}$. The dual group $\overline{G}^{\vee}$ was constructed by making a metaplectic modification of the root datum of $\mathbb{G}$.

In [70], Weissman built upon [44] and gave a construction of the "L-group" ${}^L\overline{G}$ of a *split* BD covering group $\overline{G}$. The construction in [70] is quite involved, and couched in the language of Hopf algebras. Moreover, with hindsight, it gives the correct notion only for a subclass of BD covering groups. In a foundational paper [73], Weissman gives a simpler and completely general revised construction of the L-group for an arbitrary quasi-split BD covering group (not necessarily split), using the framework of étale gerbes, thus laying the groundwork for an extension of the Langlands program to the setting of BD covering groups.

1.4. The L-group extension. — We shall describe in §4 Weissman's construction of the L-group of $\overline{G}$ for split $\mathbb{G}$ (given in the letter [69]), where one could be more down-to-earth and avoid the notion of gerbes. The fact that this more down-to-earth construction is equivalent to the more sophisticated one in [73] is shown in [72]. At this point, let us note that since $\mathbb{G}$ is split, one is inclined to simply take ${}^L\overline{G}$ as the direct product $\overline{G}^{\vee} \times W_F$, where W_F denotes the Weil group of F. At least, this is what one is conditioned to do by the theory of L-groups for linear reductive groups. However, Weissman realized that such an approach would be overly naive.

Indeed, the key insight of [70] is that the construction of the L-group of a BD covering group should be the functorial construction of an extension

$$1 \longrightarrow \overline{G}^{\vee} \longrightarrow {}^L\overline{G} \longrightarrow W_F \longrightarrow 1, \tag{1.1}$$

and an L-parameter for $\overline{G}$ should be a splitting of this short exact sequence. The point is that, *even if* ${}^L\overline{G}$ *is isomorphic to the direct product* $\overline{G}^{\vee} \times W_F$, it is not supposed to be equipped with a canonical isomorphism to $\overline{G}^{\vee} \times W_F$. This reflects the fact that there is no canonical irreducible genuine representation of $\overline{G}$, and hence there should not be any canonical L-parameter. Hence it would not be appropriate to say that the L-group of $\overline{G}$ "is" the direct product $\overline{G}^{\vee}$ with W_F.

As Weissman is the first person to make use of the full power of the Brylinski-Deligne structure theory for the purpose of representation theory and is the one who introduced the L-group extension, we shall call this evolving area the Langlands-Weissman program for BD extensions. As the adjective "evolving" is supposed to suggest, we caution the reader that the construction in [69, 73] may not be the final word on the L-group.

1.5. Results of this paper. — Against this backdrop, the purpose of this paper is to supplement the viewpoint of [70, 69, 73] concerning the L-group ${}^L\overline{G}$ in several ways. In particular, we shall study some properties of the L-group extension, suggest a variant of it and provide supporting evidence for its essential correctness. We summarize our results here: