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REGULAR POISSON MANIFOLDS OF COMPACT TYPES

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Abstract. — This is the second manuscript of a series dedicated to the study of Poisson structures of compact types (PMCTs). In this manuscript, we focus on regular PMCTs, exhibiting a rich transverse geometry. We show that their leaf spaces are integral affine orbifolds. We prove that the cohomology class of the leafwise symplectic form varies linearly and that there is a distinguished polynomial function describing the leafwise sympletic volume. The leaf space of a PMCT carries a natural Duistermaat-Heckman measure and a Weyl type integration formula holds. We introduce the notion of a symplectic gerbe, and we show that they obstruct realizing PMCTs as the base of a symplectic complete isotropic fibration (a.k.a. a non-commutative integrable system).

Résumé. **(Variétés de Poisson régulières de types compacts)** — Nous consacrons une suite d'articles aux variétés de Poisson de types compacts (nous emploierons simplement l'acronyme PMCTs). Ce travail, qui est le second de cette suite, se concentre sur les PMCTs régulières, et explore leur riche géométrie transverse. Nous montrons que l'espace de leurs feuilles sont des orbi-variétés affines entières. Nous établissons une dépendance linéaire de la classe de cohomologie de la structure symplectique dont héritent les feuilles et exhibons un polynôme qui décrit le volume symplectique des feuilles. Nous équipons l'espace des feuilles d'un PMCT d'une mesure de Duistermaat-Heckman naturelle et donnons une formule d'intégration de type Weyl. Nous introduisons enfin la notion de gerbe symplectique et montrons que celles-ci sont l'obstruction à la construction de la PMCT comme la base d'une fibration symplectique complète à fibres isotropes (autrement dit, un système intégrable non-commutatif).

CONTENTS

CHAPTER 1

INTRODUCTION

This is the second manuscript of a series of works devoted to the study of *Poisson structures of compact types* (PMCTs). These are the analogs in Poisson Geometry of compact Lie groups in Lie theory. In the first paper of this series **[13]** we have discussed general properties, described several examples, and outlined our general plan. In this paper, which is self-contained, we focus on regular PMCTs and we discover a very rich transverse geometry, where several structures, both classical and new, interact with each other in a non-trivial way. These include orbifold structures, integral affine structures, symplectic gerbes, etc. Moreover, we find that celebrated results, like the Duistermaat-Heckman Theorem on the linear variation of the symplectic class in the cohomology of reduced spaces, the polynomial behavior of the Duistermaat-Heckman measure, the Atiyah-Guillemin-Sternberg Convexity Theorem, or the Weyl Integration Formula, fit perfectly into the world of PMCTs, arising as particular statements of general results concerning PMCTs.

Given a Poisson manifold (M, π) we will look at s-connected integrations $(\mathcal{G}, \Omega)$, which are symplectic Lie groupoids of compact type. At the level of Lie groupoids, there are several compact types $\mathcal{C}$ characterized by possible conditions on $\mathcal{G}$:

$$\mathcal{C} \in \{\text{proper, s-proper, compact}\}, \tag{1.1}$$

that is, Hausdorff Lie groupoids with proper anchor map, proper source map, and compact manifold of arrows, respectively. For example, when $\mathcal{G} = G \times M$ comes from a Lie group acting on a manifold M, the three conditions correspond to the properness of the action, the compactness of G, and the compactness of both G and M, respectively. Therefore, one says that the Poisson manifold (M, π) is of:

- *$\mathcal{C}$-type* if it has an s-connected integration $(\mathcal{G}, \Omega)$ with property $\mathcal{C}$;
- *strong $\mathcal{C}$-type* if its canonical integration $\Sigma(M, \pi)$ has property $\mathcal{C}$.

A Poisson manifold (M, π) comes with a partition into symplectic leaves, generalizing the partition by coadjoint orbits from Lie theory. In this paper, we consider PMCTs where the dimension of the leaves is constant, leaving the non-regular case to the next paper in the series **[12]**. This gives rise to a regular foliation $\mathcal{F}_\pi$ on M, so, in some sense, we are looking at symplectic foliations from the perspective of Poisson Geometry.

For a general regular Poisson manifold, the leaf space

$$B = M/\mathscr{F}_\pi$$

is very pathological. However, for us, the first immediate consequence of any of the compactness conditions is that B is Hausdorff. Moreover, we will see that it comes with a very rich geometry, illustrated in the following theorem, which collects several results spread throughout the paper:

Theorem 1.0.1. — *Given a regular Poisson manifold (M,π) of proper type and an s-connected, proper symplectic integration $(\mathcal{G},\Omega)$:*

(a) *The space B of symplectic leaves comes with an* orbifold structure $\mathscr{B} = \mathscr{B}(\mathcal{G})$.
(b) *There is an induced* integral affine structure Λ *on* $\mathscr{B}$.
(c) *The classical effective orbifold underlying $\mathscr{B}$ is good.*
(d) *There is a* symplectic $\mathscr{T}$-gerbe *over $\mathscr{B}$, where $\mathscr{T}$ is the symplectic torus bundle induced by Λ. This gerbe is classified by the* Lagrangian Dixmier-Douady class*:*

$$\mathrm{c}_2(\mathcal{G},\Omega) \in H^2(\mathscr{B}, \mathscr{T}_{\mathrm{Lagr}}).$$

(e) *The class $\mathrm{c}_2(\mathcal{G},\Omega)$ vanishes if and only if (M,π) admits a* proper isotropic realization $q:(X,\Omega_X) \to (M,\pi)$ *for which $\mathcal{G} \cong \mathscr{B}_X(M,\pi)$, a natural symplectic integration constructed from X and the orbifold structure $\mathscr{B}$.*

The presence of an orbifold structure on the leaf space which, in general, is non-effective, gives rise to several technical difficulties throughout the discussion. When the symplectic leaves are 1-connected, then B is just a smooth manifold, and no further complications arise from orbifolds. In this case, all the other main features of PMCTs are already present, and it includes interesting examples, such as the regular coadjoint orbits or the principal conjugacy classes of a compact Lie group. For that reason, in the general discussion we will often consider this case first.

The different geometric structures present on the leaf space of a PMCT, mentioned in the previous theorem, interact nicely with the leafwise symplectic geometry. One illustration of this interaction is the *linear variation of symplectic forms in cohomology*, generalizing the classical Duistermaat-Heckman Theorem. For simplicity, we concentrate on the smooth case, where the leaves are 1-connected. Then to each $b \in B$ corresponds a symplectic leaf (S_b, ω_b), and the cohomologies $H^2(S_b)$ yield a bundle $\mathscr{H} \to B$. The cohomology class of the leafwise symplectic form defines a section of this bundle:

$$B \ni b \mapsto [\omega_b] \in \mathscr{H}_b = H^2(S_b).$$

In the s-proper case, the leaves are compact and $\mathscr{H}$ is a smooth flat vector bundle over B. The flat connection is the so called *Gauss-Manin connection* and arises from the underlying integral cohomology. Using parallel transport, one can compare classes $[\omega_b]$ at distinct points $b \in B$, once a path has been fixed. On the other hand, the integral affine structure on B of the previous theorem determines a developing map, defined on the universal cover of B:

$$\mathrm{dev} : \widetilde{B} \to \mathbb{R}^q \quad (q = \dim B).$$

Denoting the Chern classes of the principal torus bundle $t : s^{-1}(x_0) \to S_{b_0}$, where s and t are the source/fiber of the s-proper integration, by

$$c_1, \dots, c_q \in H^2(S_{b_0}),$$

the linear variation theorem can be stated as follows:

Theorem 1.0.2. — *If (M, π) is a regular, s-proper Poisson manifold, with 1-connected symplectic leaves, then for any path γ in B starting at b_0 one has*

$$\gamma^*([\omega_{\gamma(1)}]) = [\omega_{b_0}] + \mathrm{dev}^1(\gamma)c_1 + \cdots + \mathrm{dev}^q(\gamma)c_q.$$

Similar formulas hold for a general Poisson manifold of s-proper type.

One can also look at volume forms instead. Assume as before that we have an s-proper integration $(\mathcal{G}, \Omega)$ of (M, π). Pushing forward the Liouville measure associated to Ω, one obtains the *Duistermaat-Heckman measure* on the leaf space:

$$\mu_{DH} \in \mathcal{M}(B).$$

On the other hand, the integral affine structure on B gives rise to another measure, $\mu_{\mathrm{Aff}} \in \mathcal{M}(B)$. The classical result on the polynomial behavior of the Duistermaat-Heckman measure is a special case of the following general result for PMCTs:

Theorem 1.0.3. — *If (M, π) is a regular Poisson manifold, with s-connected, s-proper integration $(\mathcal{G}, \Omega)$, then:*

$$\mu^{\Omega}_{\mathrm{DH}} = (\iota \cdot \mathrm{vol})^2 \mu_{\mathrm{Aff}},$$

where $\mathrm{vol} : B \to \mathbb{R}$ is the leafwise symplectic volume function and $\iota : B \to \mathbb{N}$ counts the number of connected components of the isotropy group $\mathcal{G}_x$ $(x \in S_b)$. Moreover, $(\iota \cdot \mathrm{vol})^2$ is a polynomial relative to the orbifold integral affine structure on B.

The previous theorem has an interesting version already on M, where we obtain two measures, μ^M_{DH} and $\mu^{\mathrm{Aff}}_M = \mu_M$, both induced by densities ρ^M_{DH} and ρ_M, which are invariant under all Hamiltonian flows. Our study of such invariant densities yields the following Fubini type theorem:

Theorem 1.0.4. — *If (M, π) is a regular Poisson manifold, with proper integration $(\mathcal{G}, \Omega)$, then for any $f \in C^\infty_c(M)$:*

$$\int_M f(x)\,\mathrm{d}\mu_M(x) = \int_B \left(\iota(b) \int_{S_b} f(y)\,\mathrm{d}\mu_{S_b}(y)\right) \mathrm{d}\mu_{\mathrm{Aff}}(b),$$

where μ_{S_b} is the Liouville measure of the symplectic leaf S_b, and $\iota : B \to \mathbb{N}$ is the function that for each $b \in B$ counts the number of connected components of the isotropy group $\mathcal{G}_x$ $(x \in S_b)$.

We shall see in **[12]** that a similar theorem is valid for all, including non-regular, PMCTs. This theorem includes, as a special instance, the classical Weyl Integration Formula.

The rest of this paper is organized as follows. Chapter 2 is devoted to foliations and orbifolds, recalling Haefliger's approach to transversal geometry, fixing the necessary framework, but also illustrating the various compactness properties (1.1) in the simpler context of foliations. In this chapter, the orbifold structure on the leaf space of a PMCT, stated in part (a) of Theorem 1.0.1, is shown to exist.

Chapter 3 includes some basics on Integral Affine Geometry and describes its relationship with Poisson Geometry. Besides proving part (b) of Theorem 1.0.1, we discover new Poisson invariants, the so-called *extended monodromy groups* which give rise to obstructions to s-properness, but which are interesting also for general Poisson manifolds.

Chapters 4 and Chapter 5 concern Theorem 1.0.2, on the linear variation of the cohomology class of the leafwise symplectic form. We first treat the case of smooth leaf space and then the orbifold case. Both these chapters start by revisiting the developing map for integral affine structures from a novel groupoid perspective. That allows for a global formulation, free of choices, which is more appropriate for our purposes. We also obtain a decomposition result for Poisson manifolds of s-proper type which, from the point of view of classification, indicates two types of building blocks: (i) the strong proper ones with full variation, and (ii) the ones with no variation, corresponding to symplectic fibrations over integral affine manifolds.

Chapter 6 discusses the Duistermaat-Heckman measures on PMCTs and on their leaf spaces, its relationship with the measures determined by the integral affine structures, and the interaction with the Liouville measure on the symplectic leaves, leading to proofs of Theorem 1.0.3, on the polynomial nature of the Duistermaat-Heckman measure, and the integration formula of Theorem 1.0.4.

Chapter 7 explains the relationship between PMCTs and proper isotropic realizations, which appears in part (e) of Theorem 1.0.1. For any proper isotropic realization $q : (X, \Omega_X) \to (M, \pi)$ we introduce a "holonomy groupoid relative to X," $\mathrm{Hol}_X(M, \pi)$, which is usually smaller than the canonical integration $\Sigma(M, \pi)$, and hence has better chances to be proper. The groupoids $\mathrm{Hol}_X(M, \pi)$ not only arise in many examples, but are an important concept. Indeed, recall that foliations come with two standard s-connected integrations: the largest one which is the monodromy groupoid $\mathrm{Mon}(M, \mathscr{F})$ and the smallest one which is the holonomy groupoid $\mathrm{Hol}(M, \mathscr{F})$. In Poisson geometry, the integration $\Sigma(M, \pi)$ is the analog of $\mathrm{Mon}(M, \mathscr{F})$ but, in general, there is no analog of the holonomy groupoid. Our results suggest that, in Poisson Geometry, instead of looking for the smallest integration, one should look for the smallest one that acts on a given symplectic realization. This property characterizes $\mathrm{Hol}_X(M, \pi)$ uniquely.

Chapters 8 and 9 describe our theory of *symplectic gerbes*, first in the smooth case and then in the orbifold case, proving parts (d) and (e) of Theorem 1.0.1. Our departure point is the usual theory of $\mathbb{S}^1$-gerbes, which we first extend to $\mathscr{T}$-gerbes,