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PERTURBATION OF C^1-DIFFEOMORPHISMS AND GENERIC CONSERVATIVE DYNAMICS ON SURFACES

by

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***Abstract.* —** We give a survey on the main dynamical properties satisfied by C^1-generic surface diffeomorphisms. We state a connecting lemma for pseudo-orbits obtained in collaboration with M.-C. Arnaud and C. Bonatti. We then explain the perturbation techniques in C^1 topology.

***Résumé* (Perturbations des difféomorphismes C^1 et dynamique générique conservative sur les surfaces)**

Nous donnons un panorama des principales propriétés dynamiques satisfaites par les difféomorphismes C^1-génériques sur les surfaces. Nous énonçons un lemme de connexion pour les pseudo-orbites. Puis, nous expliquons les techniques de perturbations pour la topologie C^1.

0. Introduction

When one studies a mechanical system with no dissipation, the motion is governed by some ordinary differential equations which preserve a volume form. As an example, the forced damped pendulum with no friction gives rise to a conservative dynamics on the open annulus (see [**Hub99**]): let θ be the angle between the axis of a rigid pendulum with the vertical and $\dot\theta$ be the angle velocity. The configurations $(\theta,\dot\theta)$ live on the infinite annulus $\mathbb{A}=\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ and the motion is governed by the equation

$$\frac{d^2}{dt^2}\,\theta=-\sin(2\pi\theta)+h(t),$$

where h is the forcing. The volume form $d\theta\wedge d\dot\theta$ is preserved. By integrating the system, one obtains a global flow $(\phi_t)_{t\in\mathbb{R}}$, that is a family of diffeomorphisms of $\mathbb{A}$ which associates to any initial configuration $(\theta,\dot\theta)$ at time 0 the configuration $\phi_t(\theta,\dot\theta)$ at time t. If the forcing h is T-periodic, the flow satisfies the additional relation $\phi_{t+T}=\phi_t\circ\phi_T$ and the dynamics of the pendulum can be studied through the iterates

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of the conservative annulus diffeomorphism $f = \phi_T$. Several simple questions may be asked about this dynamics:

- *What are the regions $U \subset \mathbb{A}$ that are* invariant *by f?* Invariant means that $f(U) = U$.
- *Does there exist a dense set of initial data $(\theta, \dot\theta)$ which are periodic?* Periodic means that $f^\tau(\theta, \dot\theta) = (\theta, \dot\theta)$ for some integer $\tau \geq 1$.
- *How do the orbit separate?* More precizely, let us consider two initial data $(\theta, \dot\theta)$ and $(\theta', \dot\theta')$ that are close. How does the distance $d(f^n(\theta, \dot\theta), f^n(\theta, \dot\theta))$ behaves when n increases?

Another example of conservative diffeomorphism is the standard map on the two-torus:

$$(x, y) \mapsto \big(x + y, y + a\sin(2\pi(x + y))\big) \mod \mathbb{Z}^2.$$

It is sometimes considered by physicists as a model for chaotic dynamics: the equations defining such a diffeomorphism are simple but we are far from being able to give a complete description of its dynamics. However one can hope that *some other systems, arbitrarily close to the original one, could be much easier to be described.* To reach this goal, one has to precise what "arbitrarily close" and "be described" mean: the answer to our problem will depend a lot on these two definitions. The viewpoint we adopt in this text allows us to give a rather deep description of the dynamics. However one should not forget that one can choose other definitions which could seem also (more?) relevant and that very few results were obtained in this case.

The setting, Baire genericity.— In the following we consider a compact and boundaryless smooth connected surface M endowed with a smooth volume v (which is, after normalization, a probability measure) and we fix a diffeomorphism f on M which preserves v. We are aimed to describe the space of orbits of f and in particular the space of periodic orbits.

Lots of difficulties appear if one chooses an arbitrary diffeomorphism. Our philosophy here will be to forget the dynamics which seem too pathological hoping that the set of diffeomorphisms that we describe is large (at least dense in the space of dynamical systems we are working with). For us, such a set will be large if it is generic in the sense of Baire category.

This notion requires to choose carefully the space of diffeomorphisms, that should be a Baire space: for example for any $k \in \mathbb{N}$, the space Diff^k_v of C^k-diffeomorphisms of M which preserve v. A set of diffeomorphisms is *generic* (or *residual*) if it contains a dense G_δ subset of Diff^k_v, i.e. by Baire theorem if it contains a countable intersection of dense open sets of Diff^k_v (so that the intersection of two generic sets remains generic). In the sequel, we are interested in exhibiting generic properties of diffeomorphisms: these are properties that are satisfied on a generic set of diffeomorphisms.

An example: generic behavior of periodic orbits.— Robinson has proven in [**Rob70, Rob73**] the following generic property which extends a previous result of Kupka and Smale to the conservative diffeomorphisms. It is a consequence of Thom's transversality theorem.

***Theorem 0.1* (Robinson).** — *When $k \geq 1$, for any generic diffeomorphism $f \in \mathrm{Diff}^k_v$, and any periodic orbit $p, f(p), \ldots, f^\tau(p) = p$, one of the two following cases occurs (figure 1):*

- *either the orbit of p is* elliptic*: the eigenvalues of $\mathrm{D}_p\, f^\tau$ are non-real (in particular, this tangent map is conjugate to a rotation);*
- *or p is a* hyperbolic saddle*: the eigenvalues are real and have modulus different from 1. In this case, there are some one-dimensional invariant manifolds (one stable $W^s(p)$ and one unstable $W^u(p)$) through p. Points on the stable manifold converge towards the orbit of p in the future, and the same for points on the unstable manifold in the past.*

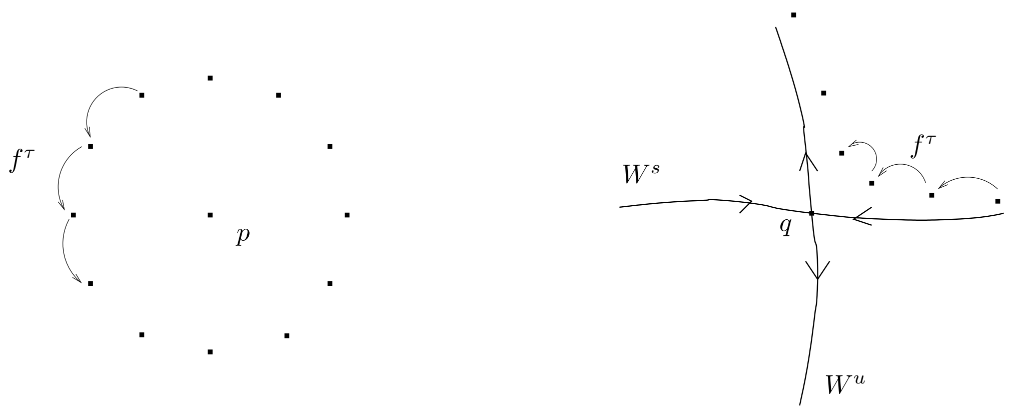

FIGURE 1. Dynamics near an elliptic point, p, and a saddle point, q.

One wants to say that the dynamics of the return map f^τ near a τ-periodic p "looks like" the dynamics of the tangent map $\mathrm{D}_p\, f^\tau$. This is the case if p is hyperbolic: D. Grobman and P. Hartman have shown that if p is hyperbolic, the map f^τ is topological conjugate to $\mathrm{D}\, f^\tau$ near p; more precizely, there exist a neighborhood U of p and V of 0 in the tangent space T_pM and a homeomorphism $h\colon U \to V$ such that $h \circ f^\tau = \mathrm{D}_p\, f^\tau \circ h$.

Theorem 0.1 implies in particular:

***Corollary 0.2*.** — *When $k \geq 1$, for any generic diffeomorphism $f \in \mathrm{Diff}^k_v$, and any $\tau \in \mathbb{N} \setminus \{0\}$, the set of periodic points of period τ is finite.*

In the next part we will give other examples of generic properties. They are often obtained in the same way:

- One first proves a *perturbation result*, which is in general the difficult part. In the previous example, one shows that for every integers $\tau, k \geq 1$, any C^k-diffeomorphism f can be perturbed in the space Diff^k_v as a diffeomorphism g whose periodic orbits of period τ are elliptic or hyperbolic.

– One then uses Baire theorem for getting the genericity. An example of this standard argument is given at section I.3.1.

The last two parts are devoted to some important perturbation results. In part II, we discuss Pugh's closing lemma that allows to create periodic orbits and Hayashi's closing lemma that allows to glue two half orbits together; the perturbations in these two cases are local. In part III, we state a connecting lemma for pseudo-orbits obtained with M.-C. Arnaud and C. Bonatti through global perturbations and explain the main ideas of its proof.

I. Overview of genericity results on the dynamics of C^1 conservative surface diffeomorphisms

This part is a survey of the properties satisfied by the C^1-generic conservative diffeomorphisms of compact surfaces

I.1. Discussions on the space Diff^k_v.

— In general, the generic properties depend strongly on the choice of the space Diff^k_v. We will here illustrate this on an example and explain why we will focus on the C^1-topology.

I.1.1. Some generic properties in different spaces. — One of the first result was given by Oxtoby and Ulam [**OU41**], in the C^0-topology.

***Theorem I.1* (Oxtoby-Ulam).** — *For any generic homeomorphism $f \in \mathrm{Diff}^0_v$, the invariant measure v is ergodic.*

Ergodicity means that for the measure v, the system cannot be decomposed: any invariant Borel set A has either measure 0 or 1. By Birkhoff's ergodic theorem, *the orbit of v-almost every point is equidistributed in M.*
The C^0 topology also seems very weak: one can show that since for any generic diffeomorphism, once there exists a periodic point of some period p, then the set of p-periodic points is uncountable. (In particular corollary 0.2 does not hold for Diff^0_v.)

In high topologies, one gets Kolmogorov-Arnold-Moser theory. One of the finest forms is given by Herman (see [**Mos73**, section II.4.c], [**Her83**, chapitre IV] or [**Yoc92**]).

***Theorem I.2* (Herman).** — *There exists a non-empty open subset $\mathcal{U}$ of Diff^k_v, with $k \geq 4$ and for any diffeomorphism $f \in \mathcal{U}$, there exists a smooth closed disk $D \subset M$ which is periodic by f: the disks D, $f(D)$,..., $f^{\tau-1}(D)$ are disjoint and $f^{\tau}(D) = D$.*

Theses disks are obtained as neighborhoods of the elliptic periodic orbits. The dynamics in this case is very different from the generic dynamics in Diff^0_v since the existence of invariant domains breaks down the ergodicity of v: the orbit of any point of D cannot leave the set $D \cup f(D) \cup \cdots \cup f^{\tau-1}(D)$.

***Remark I.3*.** — We should notice that by a result of Zehnder [**Zeh76**] for each $k \geq 1$, the C^{∞}-diffeomorphisms are dense in Diff^k_v. Therefore, for any $1 \leq k < \ell$, any property that is generic in Diff^ℓ_v will be dense in Diff^k_v. This result is not known is this generality in higher dimensions for conservative diffeomorphisms.

I.1.2. An elementary perturbation lemma. — The reason why theorem I.1 is true is that perturbations in Diff^0_v are very flexible: for any homeomorphism $f \in \mathrm{Diff}^0_v$ and any point $x \in M$, one can perturb f in order to modify the image of $f(x)$. More precisely, if y is close to $f(x)$, one chooses a small path γ that joints $f(x)$ to y. Pushing along γ, one can modify f as homeomorphism g so that $g(x) = y$. The homeomorphisms f and g will coincide outside a small neighborhood of γ. Hence, the C^0-norm of the perturbation is about equal to the distance between x and y.

In the space Diff^1_v, the C^1-norm of the perturbation also should be small (for example smaller than $\varepsilon > 0$) and one has to perturb f on a larger domain (in a ball of radius about $\varepsilon^{-1}\,\mathrm{d}(x,y)$). This can been seen easily from the mean value theorem: let x, y be two points and φ be a perturbation of the identity which satisfies $\varphi(x) = y$ and such that $\|\,\mathrm{D}\,\varphi - \mathrm{Id}\,\| \le \varepsilon$; then, if a point z is not perturbed by φ (i.e. $\varphi(z) = z$), we get

$$\|y - x\| = \|(\varphi(x) - x) - (\varphi(z) - z)\| \le \varepsilon.\|z - x\|.$$

As a consequence, when ε is small, the perturbation domain has a large radius and lot of the orbits of f will be modified.

This remark will be at the root of all the genericity results that will be presented below (a more precise statement will be given at section I.1.2). It explains the difficulty of the perturbations in Diff^1_v. In higher topologies, the situation becomes much more complicated since in Diff^k_v, the radius of the perturbation domain should be at least $(\varepsilon^{-1}\,\mathrm{d}(x,y))^{\frac{1}{k}}$.

This justifies why we will now work in Diff^1_v: we need a space of *diffeomorphisms* where the *elementary perturbations* don't have a too large support.

I.2. The closing and connecting lemmas.

— From the elementary perturbation lemma, one derivates more sophisticated perturbation lemmas.

I.2.1. Pugh's closing lemma. — The first result was shown by Pugh [**Pug67b, Pug67a, PR83, Arn98**]. It allows one to create by perturbation some periodic orbit once the dynamics is recurrent. More precisely, one considers the points z whose orbit is *non-wandering*: for any neighborhood U of z, there is a forward iterate $f^n(U)$ of U (with $n \ge 1$) which intersects U.

The local perturbation result is the following (see also figure 2):

***Theorem I.4* (Closing lemma, Pugh).** — *Let f be a C^1-diffeomorphism in Diff^1_v and $z \in M$ a non-wandering point. Then, there exists a C^1-small perturbation $g \in \mathrm{Diff}^1_v$ of f such that z is a periodic orbit of f.*

I.2.2. Hayashi's connecting lemma. — We have seen that the closing lemma allows us to connect an orbit to itself. About 30 years later, Hayashi [**Hay99, WX00, Arn01**] proved a second local perturbation lemma and showed how to connect an orbit to another one (see figure 3).