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IDENTITY ISOTOPIES ON SURFACES

by

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Abstract. — We will state an equivariant foliated version of the classical Brouwer Plane Translation Theorem and will explain how to apply this result to the study of homeomorphisms of surfaces. In particular we will explain why a diffeomorphism of a closed oriented surface of genus ≥ 1 that is the time-one map of a time dependent Hamiltonian vector field has infinitely many contractible periodic orbits. This gives a positive answer in the case of surfaces to a more general question stated by C. Conley. We will state other results about linking numbers of fixed points or periodic orbits of homeomorphisms of surfaces. We will conclude this article by introducing the free brick decompositions and explaining how to use these decompositions to get the equivariant foliated version of the Brouwer Plane Translation Theorem.

***Résumé* (Étude dynamique des homéomorphismes de surface isotopes à l'identité)**
Nous allons établir une version feuilletée équivariante du théorème classique de translation plane de Brouwer. Nous expliquerons ensuite comment utiliser ce résultat pour étudier les homéomorphismes de surfaces. En particulier, nous montrerons qu'un difféomorphisme d'une surface compacte qui est le temps 1 d'une isotopie hamiltonienne admet une infinité d'orbites périodiques contractiles, obtenant ainsi une réponse positive, dans le cas des surfaces, à une conjecture plus générale de C. Conley. Nous établirons d'autres résultats sur le nombre d'enlacement des points fixes et des points périodiques d'un homéomorphisme de surface. Nous conclurons cet article en introduisant les décompositions en briques libres et expliquerons comment se prouve le théorème de Brouwer feuilleté équivariant à partir de ces décompositions.

0. Introduction

By Cauchy-Lipschitz's theorem, one knows that if X is a smooth vector field on a manifold M and z a point of M, there exists a unique smooth map $t \mapsto F_t(z)$ which is defined on an open interval containing 0, which satisfies $F_0(z) = z$, $\frac{d}{dt}F_t(z) = X(F_t(z))$ and which cannot be extended continuously on a larger interval. In the case where each map $t \mapsto F_t(z)$ is defined on the whole real line, the vector field

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is *complete*; it is the case for example if M is compact or if M is endowed with a complete Riemmanian metric and X is bounded. If X is a smooth complete vector field, the map $(t,z)\mapsto F_t(z)$ is smooth on $\mathbf{R}\times M$ and the family $(F_t)_{t\in\mathbf{R}}$ is a *flow* or a *one parameter group*: the map $t\mapsto F_t$ is a morphism from the additive group $\mathbf{R}$ into the group of smooth diffeomorphisms of M. One obtains a *continuous dynamical system.* Such a system may be reduced, sometimes, to a *discrete dynamical system* (i.e. the study of the iterates of a diffeomorphism of a manifold) if there exists a global Poincaré section.

One can define similarly a complete time dependant smooth vector field X on a manifold M: there exists a (unique) smooth map $(t,z)\mapsto F_t(z)$ on $\mathbf{R}\times M$ such that $F_0(z)=z$, $\frac{\partial}{\partial t}F_t(z)=X(t,F_t(z))$. Such systems appear naturally when an autonomous sytem is perturbed with an exterior forcing term. Each F_t is a diffeomorphism but the family $(F_t)_{t\in\mathbf{R}}$ is no more a flow. However, if X is periodic in time, of period T, one can prove that $F_{t+T}=F_t\circ F_T$ for every $t\in\mathbf{R}$: the study of our system may be reduced to the discrete dynamical system defined by F_T. In fact one obtains an autonomous vector field Y on $\mathbf{R}/T\mathbf{Z}\times M$ by writing $Y(t,z)=(\frac{\partial}{\partial t},X(t,z))$. The induced flow admits a global Poincaré section $\{0\}\times M$ and the first return map may be written $(0,z)\mapsto(0,F_T(z))$. Note that F_T is a diffeomorphism isotopic to the identity. We will be interested here in the study of such objects. As our methods will be topological we will be interested more generally in the study of homeomorphisms of M which are isotopic to the identity, the simplest case being when the homeomorphism is the time-one map of a flow. The general problem we are interested in is the following: *find results about time-one map of flows which may be extended to time-one map of isotopies.*

As explained in the introduction of this volume, periodic orbits are highly important in the study of dynamical sytems. We will mainly be interested on results about fixed and periodic points. We will see that much can be said when $\dim M=2$, that means when M is a surface, and that such extensions are usually easy to get when the map is a diffeomorphism which is "close" to the identity.

We introduce now some definitions that we will use in this text, then we give some examples to illustrate what kind of problems we are interesting in.

By an *identity isotopy* on a manifold M we mean a continuous arc

$$\begin{aligned}[0,1] &\longrightarrow \mathrm{Homeo}(M)\\ t &\longmapsto F_t\end{aligned}$$

such that $F_0=\mathrm{Id}_M$, the last set being endowed with the compact-open topology. We naturally extend this map on $\mathbf{R}$ by writing $F_{t+1}=F_t\circ F_1$.

The *trajectory* of a point z is the arc $\gamma_z: t\mapsto F_t(z)$ defined on $[0,1]$.

A fixed point z of $F=F_1$ is *contractible* if γ_z (which is a loop) is homotopic to zero. More generally a periodic point (of period q) is contractible if the loop $\gamma_z^q=\prod_{0\leq i<q}\gamma_{F^i(z)}$ obtained by concatenating trajectories is homotopic to zero. Of course the set $\mathrm{Fix}_{\mathrm{cont}}(F)\subset\mathrm{Fix}(F)$ of contractible fixed points and the set $\mathrm{Per}_{\mathrm{cont}}(F)\subset\mathrm{Per}(F)$ of contractible periodic points depend on the chosen isotopy.

The Poincaré-Hopf's formula

$$\sum_{\xi(z)=0} i(\xi, z) = \chi(M)$$

asserts that for any continuous vector field on a compact manifold with a finite number of singularities, the sum of the Poincaré indices at each singularity is equal to the Euler characteristic $\chi(M)$ of M. The natural extension to the discrete case is the Lefschetz's formula. In the particular case of a homeomorphism F which is isotopic to the identity and has a finite number of fixed points, it may be written

$$\sum_{F(z)=z} i(F, z) = \chi(M),$$

where $i(F, z)$ is the Lefschetz index at z. An improved version (the Lefschetz-Nielsen's formula) gives us the same equality by keeping only the contractible fixed points. Endowing M with a Riemannian metric and using the exponential map, one may observe that the last formula is a consequence of the first one if F is sufficiently close to the identity for the uniform topology. Indeed there exists a uniquely defined continuous vector field X such that $\exp_z(X(z)) = F(z)$, for every $z \in M$, and the Poincaré index $i(X, z)$ is equal to the Lefschetz index $i(F, z)$.

A much more difficult example is Arnold's conjecture for Hamiltonian diffeomorphisms. Let (M, ω) be a symplectic compact manifold. If H is a smooth function on M, one can define the induced Hamiltonian vector field X_H, which is the symplectic gradient of H (i.e. $\omega_z(X_H(z), Y) = T_zH(Y)$ for every $Y \in T_zM$). Every critical point of H is a singularity of X_H. So one may minimize the number of singularities of X_H by an integer n_M, equal to the minimum number of critical points that any smooth function defined on M must have. One knows, for example, that n_M is not smaller that the number of charts necessary to cover M. One can define similarly an integer $n'_M \geq n_M$ if we restrict ourselves to the Morse functions, that means functions with only nondegenerate critical points (i.e. the bilinear form T^2_zH is non degenerate if z is a critical point). Here again, n'_M is at least equal to the sum of the Betti numbers. For example:

$$\begin{cases} n_M = n'_M = 2 & \text{if } M = S^n \text{ is a sphere,} \\ n_M = 3,\ n'_M = 2 + 2g & \text{if } M \text{ is a compact oriented surface of genus } g \geq 1, \\ n_M = n+1,\ n'_M = 2^n & \text{if } M = \mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n \text{ is a torus.} \end{cases}$$

Consider now a smooth function $H : \mathbf{R} \times M \to \mathbf{R}$ periodic in time, of period 1. Define the time dependent vector field X_H, where $X_H(t, .)$ is the symplectic gradient of $H(t, .)$, and the time-one map $F = F_1$ of the identity isotopy $(F_t)_{t\in\mathbf{R}}$ naturally defined by X_H. Let us state the

***Arnold's conjecture* ([Ar1]).** — *Is the number of contractible fixed points minimized by n_M? Is it minimized by n'_M if every fixed point of F is nondegenerate?*

In the case where F is close to the identity in the C^1-topology, one can construct a generating function whose critical points correspond to the contractible fixed points and the conjecture is true. The first proven case of the conjecture is due to C. Conley and E. Zehnder [**CoZ**] for $\mathbf{T}^{2n}$, the case of the surfaces was done a little bit later (A. Floer [**Flo1**], J.-C. Sikorav [**Si**]). For the second question, a breakthrough is due to Floer [**Flo2**] who minimized the number of contractible fixed points by the sum of the Betti numbers under a special "monotone" condition on the symplectic manifold. Floer's result was generalized by H. Hofer and D. Salamon [**HofSal**] and by K. Ono [**O**] to the "weakly monotone" case. Now this minoration is known to be true for any symplectic manifold (G. Liu, G. Tian [**LiTi**], K. Fukaya, Ono [**FuO**]).

Conley conjectured that the number of contractible periodic points was infinite in the case of a torus $\mathbf{T}^{2n}$ or in the case of a surface of genus ≥ 1. It was proved in the case where every contractible fixed point is non degenerate (see D. Salamon and Zehnder [**SalZ**]). Let us explain the conjecture by looking at the simple case of the time-one map F of the flow of a Hamiltonian vector field X_H associated to a function $H : M \to \mathbf{R}$ on a compact oriented surface of genus ≥ 1 (the previous conjecture is clearly false for the sphere, think of a rigid rotation by an irrational angle mod. π). If H is constant, the map F is the identity and every point is a contractible fixed point. Otherwise fix a regular value c in the range of H and a connected component Γ_0 of $H^{-1}(\{c\})$. One knows that H is invariant by the flow of X_H, which implies that Γ_0 is a closed orbit of this flow. This orbit is embedded in a maximal open annulus $A \subset M$ foliated by other closed orbits Γ. Moreover the limit set of each end of A contains a critical point. Note that the limit set of at least one of these ends is not reduced to a point because M is not a sphere. The time-one map F is conjugate to a rotation $R_{\nu_\Gamma+\mathbf{Z}}$ on each circle Γ, where $\nu_\Gamma \in (0, +\infty)$ is well defined and depends continuously on Γ. Moreover ν_Γ tends to zero when Γ tends to each end whose limit set is not reduced to a point. One concludes that there exist periodic points of arbitrarily large period. In the particular case where H has a finite number of critical points, choose one point where H reaches its maximum. One of the previous annulus is such that the limit set of one of its ends is reduced to that point. Observe that the periodic points of F which are in this annulus are necessarily contractible. The existence of contractible periodic points of arbitrarily large period may be proven in the same way if $\mathrm{Crit}(H)$ is "topologically trivial", that means contained in a disk.

The usual way to compute n_M and n'_M is to equip M with a Riemannian metric and to study the dynamics of the gradient vector field associated to H (Morse's theory). This vector field Y is not uniquely defined (it depends on the Riemannian metric) and the induced flow is transverse to the Hamiltonian flow outside $\mathrm{Crit}(H)$, if $\dim M = 2$. Every leaf of the foliation $\mathcal{F}$ by orbits of Y is pushed on its left by the isotopy $(F_t)_{t\in\mathbf{R}}$ associated to X_H. More precisely, every orbit of X_H is positively transverse to $\mathcal{F}$, it intersects transversally every leaf, crossing it from the right to the left. We will see that *dynamically transverse foliations* can be constructed for the time-one map of a Hamiltonian identity isotopy and more generally for the time-one map F of any identity isotopy $(F_t)_{t\in[0,1]}$ in the following sense. One can find

- a (non necessarily unique) closed subset $Z \subset \mathrm{Fix}_{\mathrm{cont}}(F)$;
- an identity isotopy $(F'_t)_{t\in[0,1]}$ homotopic (relative to the ends) to $(F_t)_{t\in[0,1]}$, such that $F'_t(z) = z$, for every $z \in Z$ and every $t \in [0,1]$;
- a non singular oriented topological foliation $\mathcal{F}$ on $M \setminus Z$;

such that every leaf of $\mathcal{F}$ is pushed on its left by the isotopy $(F'_t)_{t\in[0,1]}$: every trajectory $z \mapsto F'_t(z)$ is homotopic (relative to the ends) to an arc that is positively transverse to $\mathcal{F}$.

We will obtain in that way an interesting tool to study homeomorphisms of surfaces. It will permit us to give in section 5 a proof of Conley's conjecture in the case of a compact surface of genus ≥ 1.

Let us explain now the structure of this article.

We will recall in the next section Brouwer's theory of orientation preserving homeomorphisms of the euclidean plane and in particular Brouwer's plane translation theorem. This theorem asserts that if F is a given orientation preserving and fixed point free homeomorphism of $\mathbf{R}^2$, then by any point passes a Brouwer line, that means a properly imbedded line Γ which separates its direct and inverse image by F. Brouwer's theory may be seen as an extension for orientation preserving homeomorphisms of $\mathbf{R}^2$ of some results which are obviously true in the case of time-one maps of flows on $\mathbf{R}^2$.

In section 2 we will state an improved version of Brouwer's plane translation theorem. If F is an orientation preserving and fixed point free homeomorphism of $\mathbf{R}^2$ we will foliate the plane by Brouwer lines. Moreover if F commutes with the elements of a discrete group of orientation preserving homeomorphisms, which acts freely and properly on $\mathbf{R}^2$, the foliation may be chosen invariant by the action of the group. One may reformulate this result. Let F be the time-one map of an identity isotopy on a surface, if there is no contractible fixed point then one can find a foliation dynamically transverse to the isotopy: every trajectory $z \mapsto F_t(z)$ is homotopic (relative to the ends) to an arc that is positively transverse to $\mathcal{F}$. The ideas of the proofs of these results will be given in section 6.

In section 3 we will give the first applications of the existence of dynamically transverse foliations. Let F be the time-one map of an identity isotopy $(F_t)_{t\in[0,1]}$ on a compact surface M of genus $g \geq 1$ and $\widetilde{F}$ the time-one map of the lifted identity isotopy $(\widetilde{F}_t)_{t\in[0,1]}$ on the universal covering space $\widetilde{M}$. One can define the *linking number* $I(z,z') \in \mathbf{Z}$ of two distinct fixed points of $\widetilde{F}$, equal to the winding number of $t \mapsto \widetilde{F}_t(z') - \widetilde{F}_t(z)$ (a precise definition is given in the section). Similarly, one can define the linking number $I(z,z')$ where z is a fixed point and z' a periodic point of $\widetilde{F}$. The main result of this section is the fact, which will be useful later, that for every periodic point z' of $\widetilde{F}$ of period $q \geq 2$, there exists a fixed point z such that $I(z,z') \neq 0$. Similar results on the sphere are proven in the same section.

In section 4 we will state a topological generalization of the classical Poincaré-Birkhoff's theorem about homeomorphisms of the annulus. We will give a proof by using the existence of dynamically transverse foliations. Using the results of the previous section, one will deduce that if F is the time-one map of an identity isotopy $(F_t)_{t\in[0,1]}$ on a compact surface M of genus $g \geq 1$ which preserves a probability measure with total support, and which has a contractible periodic point of period $q \geq 2$, then F has contractible periodic points of period arbitrarily large. There is a similar result on the sphere if one supposes moreover that F has at least three fixed points.