Panoramas & Synthèses
24, 2007, p. 33–67

ARC-SYMMETRIC SETS AND ARC-ANALYTIC MAPPINGS

by

Krzysztof Kurdyka & Adam Parusiński

Abstract. — Arc-symmetric subsets of a real analytic manifold are the subsets which satisfy the following test: given an analytic arc, then either the arc meets the set at isolated points or it is entirely included in the set. Arc-symmetric semialgebraic subsets of an affine space form a family which contains all connected (even analytic) components of real algebraic sets. Taking the sets of this family as closed sets we obtain a noetherian topology $\mathcal{AR}$ on $\mathbb{R}^n$, stronger than the Zariski topology. We show that the $\mathcal{AR}$ topology has similar properties to the Zariski one in the complex algebraic case, consequently we obtain some new topological methods in the real algebraic geometry. As an application we prove that injective regular self-maps of real algebraic sets are surjective, this is a real version of an analogous theorem of Ax for algebraically closed fields. We give a proof of a result of Kucharz that the homology classes with $\mathbb{Z}_2$ coefficients of compact Nash manifolds can be realised by the fundamental classes of arc-symmetric semialgebraic sets.

***Résumé* (Ensembles symétriques par arcs et applications arc-analytiques).** — Les ensembles symétriques par arcs d'une variété analytique réelle sont les sous-ensembles vérifiant la condition : un arc analytique rencontre l'ensemble uniquement en des points isolés, ou bien est entièrement contenu dans l'ensemble. Les ensembles semi-algébriques symétriques par arcs d'un espace affine forment une famille contenant toutes les composantes connexes (même celles analytiques) des ensembles algébriques réels. En prenant cette famille de parties pour collection de fermés, on obtient une topologie noethérienne $\mathcal{AR}$ sur $\mathbb{R}^n$ plus fine que la topologie de Zariski. On montre que la topologie $\mathcal{AR}$ possède des propriétés similaires à celle de Zariski dans le cas algébrique complexe. On en déduit de nouvelles méthodes topologiques en géométrie algébrique réelle. Comme application nous montrons notamment que toute application injective et régulière d'un ensemble algébrique réel dans lui-même est surjective. C'est une version réelle du théorème d'Ax du cas algébriquement clos. Nous donnons aussi une preuve d'un résultat de Kucharz : toute classe d'homologie, à coefficients dans $\mathbb{Z}_2$, d'une variété de Nash compacte, se réalise comme classe fondamentale d'un sous-ensemble semi-algébrique symétrique par arcs.

2000 *Mathematics Subject Classification.* — 14Pxx, 14P05, 14P25 , 32B10, 32C05, 58A07.
Key words and phrases. — Analytic arcs, real algebraic sets, arc-symmetric semi-algebraic sets, arc-analytic mappings, Nash constructible functions, Euler integral, injective regular selfmaps.

1. Introduction

The purpose of this paper is to introduce the reader to the technique of arc-symmetric sets and arc-analytic mappings. For reader's convenience we include some exercises in the text and also state some open questions which can motivate further research.

Arc-symmetric sets and arc-analytic functions were introduced in [**23**] by the first-named author. One of the motivation was a striking difference between real algebraic (or analytic) geometry and the geometry over the field of complex numbers or more generally over an algebraically closed field. In the complex case the topology is adequate to the algebra; for instance the irreducible sets are connected in the strong topology. Over reals the irreducible nonsingular cubic $C = \{x^3 - x = y^2\}$ has two connected components: E_c which is compact and E_n which is noncompact. Now let us take the cone $\widetilde{C} = \{x^3 - xz^2 = y^2\}$ over the curve C, geometrically we place the curve C in the plane $\{z = 1\}$ in $\mathbb{R}^3$ and we draw lines trough the origin and the points of C. We have $\widetilde{C} = \widetilde{E}_c \cup \widetilde{E}_n$, where $\widetilde{E}_c$ and $\widetilde{E}_n$ are the cones corresponding to the curves E_c and E_n. Clearly $\widetilde{C}$ is a connected irreducible algebraic subset of $\mathbb{R}^3$. Now blow up the origin in $\mathbb{R}^3$ and observe that the strict transform of $\widetilde{C}$ is just $C \times \mathbb{R} = (E_c \times \mathbb{R}) \cup (E_n \times \mathbb{R})$, so it has again two connected components and is irreducible as an algebraic set. So we see that the "components" $\widetilde{E}_c$ and $\widetilde{E}_n$ persist. In other words from geometrico-topological point of view the set $\widetilde{C}$ is not irreducible but the components we want to exhibit are finer than the connected components of $\widetilde{C}$. Actually our arc-symmetric semialgebraic sets will allow us to detect the "components" $\widetilde{E}_c$ and $\widetilde{E}_n$.

Arc-symmetric subsets of a real analytic manifold are the subsets which satisfy the following test: given an analytic (parametrized) arc, then either the arc meets the set at isolated points or it is entirely included in the set. Arc-symmetric semialgebraic subsets of an affine space form a family which contains all connected (even analytic) components of real algebraic sets. Taking the sets of this family as closed sets we define a topology $\mathcal{AR}$ on $\mathbb{R}^n$, we prove that this topology is actually noetherian, and stronger than the Zariski topology. Moreover the $\mathcal{AR}$-irreducible components are connected (and closed) for the strong topology. So, to some extent, the $\mathcal{AR}$ topology is similar to the Zariski topology in the complex case and it is well known how powerful are topological methods in the complex case.

At the beginning one of the target applications of this theory was the study of the problem: let $X \subset \mathbb{R}^n$ be an algebraic set, assume that $f : X \to X$ is regular (restriction of a polynomial) and injective, is it true that f is surjective ? It was know that the answer was positive for X complex algebraic (or more generally for X algebraic over algebraically closed field) by Ax's theorem [**2**]. Let us recall that the result of Ax was established by means of model theory. In the real case the positive answer to the question of surjectivity of injective endomorphism was obtained only in the case where X is nonsingular by A. Borel [**10**], see also [**8**]. The general case, when X is a real algebraic set (possibly singular) was solved only recently by the first-named author in [**26**], his proof uses arc-symmetric semialgebraic sets, in particular the fact that each such set carries a fundamental class. Recently the second-named author gave in [**38**] a simplified proof of the surjectivity theorem (in a slightly more general case). This proof also uses the arc-symmetric semialgebraic sets but in a more topological way and shows moreover that an injective endomorphism is a homeomorphism. In chapter 5 we give a full proof of the surjectivity theorem.

In the second chapter we recall some basic facts about analytic arcs, their parametrizations and equivalencies. We prove the main results (mentioned above) on arc-symmetric semialgebraic sets. We recall also the notion of arc-analytic function introduced in [**23**]. Let M, N be real analytic manifolds, we say that a function $f : M \to N$ is *arc-analytic* if $f \circ \gamma$ is analytic for every analytic arc γ. Clearly the zero sets of arc-analytic functions are arc-symmetric. Arc-analytic maps are natural morphisms for the category of arc-symmetric sets, actually they are closely related to blow-analytic maps introduced by T.C. Kuo in 70's. A mapping $f : M \to N$ is called *blow-analytic* if there exists $\sigma : \widetilde{M} \to M$ a locally finite composition of blowing-ups with smooth centers such that $f \circ \sigma$ is analytic. Clearly each blow-analytic map is arc-analytic (and subanalytic). It was conjectured by K. Kurdyka and shown by Bierstone and Milman [**5**], see also [**36**], that the converse is true in the semialgebraic category. The full converse statement in the general (subanalytic case) remains an open problem, where only some partial result based on the local blowing-up procedure is known, cf. [**5**] and [**36**].

In chapter 3 we study the local topological properties of arc-symmetric sets. We use the technique of the Euler integral of constructible functions, in a similar way as for the real algebraic sets as explained in [**11**]. In particular we show that the arc-symmetric sets satisfy the same local topological properties as the real algebraic ones, they are (mod 2) Euler spaces, for instance. This similarity is more transparent if we restrict ourselves to the compact arc-symmetric sets or, in general, to the finite set-theoretic combinations of compact arc-symmetric sets, called the $\mathcal{AS}$-sets. For such sets we do not need the properness assumption to show that the image of an $\mathcal{AS}$-set by an injective map with an $\mathcal{AS}$ graph is again an $\mathcal{AS}$-set. This is crucial for the proof of surjectivity of injective endomorphisms.

In chapter 3 we recall also a result of Kurdyka and Rusek [**27**] which says that $\pi_{n-d-1}(\mathbb{R}^n \setminus X) \neq \{0\}$ for X arc-symmetric and semialgebraic of dimension d. It implies easily theorem of Białynicki-Birula and Rosenlicht [**4**] about surjectivity of injective polynomial maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

In chapter 4 we present a formal axiomatic approach to the $\mathcal{AS}$ sets and the algebraically constructible sets (the finite set-theoretic combinations of real algebraic sets) based on the notion of constructible categories of [**38**].

In chapter 5 we propose two fast applications of arc-symmetric sets. Firstly we give a topological proof of surjectivity theorem. Secondly we give a proof of a recent observation by W. Kucharz [**21**] and K. Kurdyka that the homology classes with $\mathbb{Z}_2$ coefficients of compact Nash manifolds can be realised by the fundamental classes of arc-symmetric semialgebraic sets (recall that this is false if we require these sets to be algebraic).

Recently the arc-symmetric sets appeared as well in the construction of new invariants in the singularity theory in conjuction with another theory describing arc spaces: the motivic integration. Recall that for the real algebraic sets new additive invariants, called the virtual Betti numbers, have been proposed in [**32**]. These invariants have been extended to the $\mathcal{AS}$-sets by Fichou in [**13**]. The additivity of this invariants allow one to construct new invariants of real analytic function germs, analogous to the zeta function of Denef and Loeser, that are used to classify these germs with respect to the blow-analytic and the blow-Nash equivalence. We refer the reader to [**20**], [**13**], [**14**], and [**17**] of this volume, for details. It is our strong conviction that the arc-symmetric sets and the arc-analytic mapping give a natural framework in which many ideas and techniques of motivic integration can be adapted to the real algebraic and analytic geometry, and the study of real analytic singularities.

2. Arc-symmetric subsets of affine space

2.1. Analytic arcs. — By *an analytic arc* in an analytic manifold M we mean an analytic non-constant mapping $\gamma : (-\varepsilon, \varepsilon) \to M$, for some $\varepsilon > 0$. The image of the germ of γ at 0 is well-defined. Indeed, γ as a mapping to a neighborhood of $\gamma(0)$, is finite and proper, (cf. [**28**]). The image of the germ of γ may be of two topological types.

If γ is injective, then the image of the germ of γ is an irreducible analytic germ of dimension 1, which is homeomorphic to the germ of the interval $(-\alpha, \alpha)$ at 0. If γ is not injective (in a neighborhood of 0), then, as we prove below, $\gamma(t) = \eta(t^k)$, where $k = 2^d$ and η is an analytic injective arc. So in this case the image of the germ of γ is homeomorphic to the germ of $[0, \alpha)$ at 0, it is a half-branch of an irreducible analytic germ of dimension 1. What is crucial for our theory is the fact that one half-branch determines the other.

Later on we shall not distinguish between an arc and its germ at 0. In the sequel we will work rather with the images of analytic arcs (by abuse of language we will call them sometimes analytic arcs as well). To explain when two injective analytic arcs have the same image we define an equivalence relation on the space of all (not necessarily injective) analytic arcs.

We say that two arcs γ and γ' are *equivalent* if there exists η a germ of analytic arc, and h, h' two germs of analytic homeomorphisms of neighborhoods of $0 \in \mathbb{R}$, such that

$$\gamma = \eta \circ h \quad \text{and} \quad \gamma' = \eta \circ h'. \tag{1}$$

For instance $\gamma(t) = (t^2, t^3)$ and $\gamma' = (t^6, t^9)$ are equivalent, where $\eta = \gamma$, $h(t) = t$, and $h'(t) = t^3$. Note that, in general, an analytic homeomorphism has not an analytic inverse. This is an equivalence relation, but the transitivity is not obvious.

Since we will work locally we may assume that $M = \mathbb{R}^n$. If $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is an analytic arc, that is the components of γ are convergent power series, then $z \mapsto \gamma(z)$ is a well defined holomorphic function on $\{z \in \mathbb{C} : |z| < \varepsilon'\}$, for some $\varepsilon \geq \varepsilon' > 0$. Note that $\gamma(z)$ is a holomorphic function which is *real*, that is $\overline{\gamma(\bar{z})} = \gamma(z)$. We state now a classical fact about germs of complex analytic arcs. Recall that the image of such an arc is homeomorphic to a disc.

Lemma 2.1. — *Let $\gamma : \{z \in \mathbb{C}; |z| < \varepsilon\} \to \mathbb{C}^n$ be holomorphic and non-constant. Then there exists an integer $k \geq 1$ and $\eta : \{z \in \mathbb{C}; |z| < \delta\} \to \mathbb{C}^n$ an injective holomorphic mapping such that, after a holomorphic change of variable in a neighborhood of $0 \in \mathbb{C}$, we have*

$$\gamma(z) = \eta(z^k).$$

The integer k is uniquely determined. Moreover, if γ is real, then η and the change of variables can be chosen real.

We give a sketch of the proof. Let us assume that $\gamma(0) = 0$ and write $\gamma = (\gamma_1, \ldots, \gamma_n)$, where $\gamma_i : \{|z| < \varepsilon\} :\to \mathbb{C}$ are holomorphic functions. After a suitable permutation of coordinates in $\mathbb{C}^n$ we may assume that the orders of γ_i's are increasing. In particular γ_1 is nonconstant so we can write $\gamma_1(z) = z^\ell u(z)$, with some integer ℓ and u holomorphic function, $u(0) \neq 0$. Note that $w = zu(z)^{1/\ell}$ defines a holomorphic change of variable in a neighborhood of $0 \in \mathbb{C}$. Observe that this change of variable is real if γ_1 is real. Clearly $\gamma_1(w) = w^\ell$. So we may assume that

$$\gamma(z) = \big(z^\ell, \gamma_2(z), \ldots, \gamma_n(z)\big),$$

where $\gamma_i(z) = \sum_{\nu=1}^{\infty} a_\nu^i z^\nu$ are convergent power series. Let k be the greatest common divisor of ℓ and the set of all $\nu \in \mathbb{N}$ such that $a_\nu^i \neq 0$ for some $i \in \{2, \ldots, n\}$. Thus we can write $\gamma(z) = \eta(z^k)$, where

$$\eta(z) = \big(z^{\ell/k}, \eta_2(z), \ldots, \eta_n(z)\big) \quad \text{and} \quad \eta_i(z) = \sum_{\nu=1}^{\infty} a_\nu^i z^{\nu/k}.$$

But now the greatest common divisor of exponents of all powers which appear effectively in η is equal to 1, hence η must be injective (see e.g. Walker [**45**], chap. IV, thm. 2.1). Clearly, if γ is real, then η is real as well.

Remark 2.2. — Note that in lemma 2.1 the injective parametrization η is determined up to composition with a biholomorphism of a neighborhood of $0 \in \mathbb{C}$. We will call such a parametrization *irreducible.*