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ALGEBRAICALLY CONSTRUCTIBLE FUNCTIONS: REAL ALGEBRA AND TOPOLOGY

by

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Abstract. — Algebraically constructible functions connect real algebra with the topology of algebraic sets. In this survey we present some history, definitions, properties, and algebraic characterizations of algebraically constructible functions, and a description of local obstructions for a topological space to be homeomorphic to a real algebraic set.

***Résumé* (Fonctions algébriquement constructibles : algèbre réelle et topologie)**
Les fonctions algébriquement constructibles établissent un lien entre l'algèbre réelle et la topologie des ensembles algébriques réels. Dans cet article on présente l'historique, les définitions, les propriétés basiques et des caractérisations algébriques des fonctions algébriquement constructibles, ainsi qu'une description de l'obstruction locale pour qu'un espace topologique soit homéomorphe à un ensemble algébrique réel.

More than three decades ago Sullivan proved that the link of every point in a real algebraic set has even Euler characteristic. Related topological invariants of real algebraic singularities have been defined by Akbulut and King using resolution towers and by Coste and Kurdyka using the real spectrum and stratifications.

Sullivan's discovery was motivated by a combinatorial formula for Stiefel-Whitney classes. Deligne interpreted these classes as natural transformations from constructible functions to homology. Constructible functions have interesting operations inherited from sheaf theory: sum, product, pullback, pushforward, duality, and integral. Duality is closely related to a topological *link operator.* To study the topology of algebraic sets the authors introduced *algebraically constructible functions.* Using the link operator we have defined many local invariants which generalize those of Akbulut-King and Coste-Kurdyka.

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Algebraically constructible functions are interesting from a purely algebraic viewpoint. From the theory of basic algebraic sets it follows that if a constructible function φ on an algebraic set of dimension d is divisible by 2^d then φ is algebraically constructible. Parusiński and Szafraniec showed that algebraically constructible functions are precisely those constructible functions which are sums of signs of polynomials. Bonnard has given a characterization of algebraically constructible functions using fans, and she has investigated the number of polynomials necessary to represent an algebraically constructible function as a sum of signs of polynomials. Pennaneac'h has developed a theory of *algebraically constructible chains* using the real spectrum.

In section 1 we briefly discuss the results of Sullivan, Akbulut-King, and Coste-Kurdyda. In the next section we define algebraically constructible functions and their operations. In section 3 we discuss the relations of algebraically constructible functions with real algebra. In the following section we describe how to generate our local topological invariants. In the final section we raise some questions for future research. Throughout we consider only algebraic subsets of affine space.

This paper was originally written for the 2001 meeting in Rennes of the *Real Algebraic and Analytic Geometry Network* (RAAG), and it was posted on the RAAG website as a state-of-the-art survey. Related survey articles have been written recently by Coste [**15**], [**13**], Bonnard [**8**], and McCrory [**26**]. We thank Michel Coste for his encouragement and insight.

1. Akbulut-King numbers

Let X be a real semialgebraic set in $\mathbb{R}^n$, and let $x \in X$. Let $S(x, \varepsilon)$ be the sphere of radius $\varepsilon > 0$ in $\mathbb{R}^n$ centered at x. By the local conic structure lemma [**6**, (9.3.6)], for ε sufficiently small the topological type of the space $S(x, \varepsilon) \cap X$ is independent of ε. This space is called the *link* of x in X, and it is denoted by $\operatorname{lk}(x, X)$.

Our starting point is Sullivan's theorem [**35**]:

Theorem 1.1. — *If X is a real algebraic set in $\mathbb{R}^n$ and $x \in X$ then the Euler characteristic $\chi(\operatorname{lk}(x, X))$ is even.*

For example, the "theta space" $X \subset \mathbb{R}^2$,

$$X = \{(x, y) \mid x^2 + y^2 = 1\} \cup \{(x, y) \mid -1 \le x \le 1,\ y = 0\},$$

is not homeomorphic to an algebraic set, for the link of the point $(1, 0)$ (or the point $(-1, 0)$) in X is three points, which has odd Euler characteristic.

Many proofs of Sullivan's theorem have been published; see [**12**], [**20**], [**5**, (3.10.4)], [**6**, (11.2.2)], [**18**, (4.4)]. Sullivan's original idea was to use complexification. First he proved that the link of x in the complexification $X_{\mathbb{C}}$ has Euler characteristic 0, and then he used that $\operatorname{lk}(x, X)$ is the fixed point set of complex conjugation on $\operatorname{lk}(x, X_{\mathbb{C}})$

to deduce that
$$\chi\big(\operatorname{lk}(x,X)\big) \equiv \chi\big(\operatorname{lk}(x,X_{\mathbb{C}})\big) \pmod 2.$$
Mather [**25**, p. 221] gave a proof that the link L of a point in a complex algebraic set has Euler characteristic 0 by constructing a tangent vector field on L which integrates to a nontrivial flow of L.

The following result puts Sullivan's theorem in a more general context (*cf.* [**3**, (2.3.2)]).

Theorem 1.2. — *If X and Y are real algebraic sets with Y irreducible and $f : X \to Y$ is a regular map, there is an algebraic subset Z of Y with $\dim Z < \dim Y$ such that the Euler characteristic $\chi(f^{-1}(y))$ is constant mod 2 for $y \in Y \setminus Z$.*

In other words, the Euler characteristic is generically constant mod 2 in every family of real algebraic sets. To deduce Sullivan's theorem as a corollary let $Y = \mathbb{R}$, $x_0 \in X$, and $f(x) = (x - x_0)^2$. For $y < 0$ the fiber $f^{-1}(y)$ is empty, and for $y > 0$ sufficiently small, the fiber $f^{-1}(y)$ is $\operatorname{lk}(x_0, X)$.

Benedetti-Dedò [**4**] and Akbulut-King [**2**] proved that Sullivan's condition is not only necessary but also sufficent in low dimensions: If X is a compact triangulable space of dimension less than or equal to 2, and the link of every point has even Euler characteristic, then X is homeomorphic to a real algebraic set. (The link of a point in a triangulable space is the boundary of a simplicial neighborhood.) A triangulable space such that the link of every point has even Euler characteristic is called an *Euler space*.

Akbulut and King [**3**] showed that the situation in dimension 3 is more complicated. They defined four non-trivial topological invariants of a compact Euler space Y of dimension at most 2, $a_i(Y) \in \mathbb{Z}/2$, $i = 0, 1, 2, 3$ (with $a_i(Y) = 0$ when $\dim Y < 2$). Let $\chi_2(Y)$ be the Euler characteristic mod 2. It is easy to see that if X is an Euler space then the link of every point of X is an Euler space.

Theorem 1.3. — *A compact 3-dimensional triangulable topological space X is homeomorphic to a real algebraic set if and only if, for all $x \in X$,*
$$\chi_2\big(\operatorname{lk}(x,X)\big) = 0 \quad \textit{and} \quad a_i\big(\operatorname{lk}(x,X)\big) = 0, \quad i = 0,1,2,3.$$

Akbulut and King's invariants arise from a combinatorial analysis of the resolution of singularities of an algebraic set. The elementary definition of these *Akbulut-King numbers* and computations of examples can be found in Akbulut and King's monograph [**3**, chap. VII, pp. 190–197]. (In the terminology of [**3**, (7.1.1)], $a_i(\operatorname{lk}(x,X))$ is the mod 2 Euler characteristic of the link of x in the characteristic subspace $\mathcal{Z}_i(X)$.) The depth of the method of *resolution towers* is shown by the remarkable result that the vanishing of the Akbulut-King numbers gives a sufficient condition for a triangulable 3-dimensional space to be homeomorphic to an algebraic set. Chapter I of [**3**] is an introduction to their methods, with informative examples.

Another descendant of Sullivan's theorem is due to Coste and Kurdyka [**16**]:

Theorem 1.4. — *Let X be an algebraic set and let V be an irreducible algebraic subset. For $x \in V$ the Euler characteristic of the link of x in X is generically constant mod 4: There is an algebraic subset W of V with $\dim W < \dim V$ such that $\chi(\mathrm{lk}(x, X))$ is constant mod 4 for $x \in V \setminus W$.*

This theorem was first proved by Coste [**14**] when $\dim X - \dim V \leq 2$ using chains of specializations of points in the real spectrum. The general case was proved topologically using stratifying families of polynomials. It can also be proved using Akbulut and King's topological resolution towers (see [**3**], exercise on p. 192).

Using the same techniques Coste and Kurdyka defined invariants mod 2^k associated to chains $X_1 \subset X_2 \subset \cdots \subset X_k$ of algebraic subsets of X (see [**16**, thm. 4]). Furthermore they used their mod 4 and mod 8 invariants to recover the Akbulut-King numbers. Using a relation between complex conjugation and the monodromy of the complex Milnor fibre of an ordered family of functions, the authors [**28**] reinterpreted and generalized the Coste-Kurdyka invariants as Euler characteristics of iterated links.

2. Constructible Functions

Algebraically constructible functions were introduced by the authors [**27**] as a vehicle for using the ideas of Coste and Kurdyka to generate new Akbulut-King numbers.

Let X be a real semialgebraic set. A *constructible function* on X is an integer-valued function

$$\varphi : X \longrightarrow \mathbb{Z}$$

which can be written as a finite sum

$$\varphi = \sum m_i \mathbf{1}_{X_i}, \tag{1}$$

where for each i, X_i is a semialgebraic subset of X, $\mathbf{1}_{X_i}$ is the characteristic function of X_i, and m_i is an integer.

The set of constructible functions on X is a ring under pointwise *sum* and *product.* If $f : X \to Y$ is a semialgebraic map and φ is a constructible function on Y, the *pullback* $f^*\varphi$ is the constructible function defined by

$$f^*\varphi(x) = \varphi\big(f(x)\big). \tag{2}$$

The operations of pushforward and duality are defined using the Euler characteristic. If φ has compact support one may assume that the sets X_i in (1) are compact, and the *Euler integral* is defined by

$$\int_X \varphi \,\mathrm{d}\chi = \sum m_i \chi(X_i). \tag{3}$$

The Euler integral is additive, and it does not depend on the choice of representation (1) of φ.

If $f : X \to Y$ is a proper semialgebraic map and φ is a constructible function on X, the *pushforward*, $f_*\varphi$ is the constructible function on Y given by

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi \,\mathrm{d}\chi. \tag{4}$$

Suppose that X is a semialgebraic set in $\mathbb{R}^n$. If φ is a constructible function on X, the *link* $\Lambda\,\varphi$ is the constructible function on X defined by

$$\Lambda\varphi(x) = \int_{S(x,\varepsilon)\cap X} \varphi \,\mathrm{d}\chi, \tag{5}$$

for $\varepsilon > 0$ sufficiently small.

The *dual* $D\varphi$ is defined by

$$D\varphi = \varphi - \Lambda\,\varphi. \tag{6}$$

The operations sum, product, pullback, pushforward, and dual come from sheaf theory. Operations on constructible functions have been studied by Kashiwara and Schapira [**21**], [**34**] and by Viro [**36**].

Now suppose that X is a real algebraic set. A provisional definition of algebraically constructible functions would be to require the sets X_i in (1) to be algebraic subsets of X. But the image of an algebraic set by a proper regular map is not necessarily algebraic, so this class of functions – which we call *strongly algebraically constructible* – is not preserved by the pushforward operation. To remedy this defect we make the following definition.

Let X be a real algebraic set. An *algebraically constructible function* on X is an integer-valued function which can be written as a finite sum

$$\varphi = \sum m_i f_{i*} \mathbf{1}_{Z_i}, \tag{7}$$

where for each i, Z_i is an algebraic set, $\mathbf{1}_{Z_i}$ is the characteristic function of X_i, $f_i : Z_i \to X$ is a proper regular map, and m_i is an integer.

Clearly the sum of algebraically constructible functions is algebraically constructible. The product of algebraically constructible functions is algebraically constructible because the fiber product of algebraic sets over X is an algebraic set over X: If $f_1 : Z_1 \to X$ and $f_2 : Z_2 \to X$ are proper regular maps, then so is the fiber product $f : Z_1 \times_X Z_2 \to X$,

$$\begin{array}{ccc} Z_1 \times_X Z_2 & \longrightarrow & Z_2 \\ \downarrow & & \downarrow{\scriptstyle f_2} \\ Z_1 & \xrightarrow{f_1} & X \end{array}$$

where $Z_1 \times_X Z_2 = \{(z_1, z_2) \mid f_1(z_1) = f_2(z_2)\}$ and $f(z_1, z_2) = f_1(z_1) = f_2(z_2)$. Furthermore, for all $x \in X$, $f^{-1}(x) = f_1^{-1}(x) \times f_2^{-1}(x)$.