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DISTORTION IN GROUPS OF CIRCLE AND SURFACE DIFFEOMORPHISMS

by

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Abstract. — If G is a finitely generated group with generators $\{g_1, \ldots, g_j\}$ then an infinite order element $f \in G$ is a *distortion element* of G provided $\liminf_{n\to\infty} |f^n|/n = 0$, where $|f^n|$ is the word length of f^n in the generators. We survey a number of results concerning this concept and its application to group actions on surfaces, especially those which preserve a Borel measure. Let S be a closed orientable surface and let $\mathrm{Diff}(S)_0$ denote the identity component of the group of C^1 diffeomorphisms of S. One of the results we discuss asserts that if S has genus at least two and if f is a distortion element in some finitely generated subgroup of $\mathrm{Diff}(S)_0$, then $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$ for every f-invariant Borel probability measure μ. We also compare results for surface diffeomorphisms with analogous results for the circle.

***Résumé* (Distorsion pour les groupes de difféomorphismes du cercle et des surfaces)**

Si G est un groupe finiment engendré par des générateurs $\{g_1, \ldots, g_j\}$, un élément d'ordre infini $f \in G$ est un *élément de distorsion* de G lorsque $\liminf_{n\to\infty} |f^n|/n = 0$, où $|f^n|$ est la longueur de f^n comme mot en les générateurs. Nous présentons un panorama de nombreux résultats autour de cette notion et donnons des applications aux actions de groupes sur les surfaces, notamment dans le cas où une mesure de Lebesgue est préservée. Soit S une surface fermée orientable et soit $\mathrm{Diff}(S)_0$ la composante de l'identité dans le groupe des difféomorphismes de classe C^1 de S. Un des résultats présentés affirme que si le genre de S est au moins égal à 2, et si f est un élément de distorsion dans un sous-groupe de $\mathrm{Diff}(S)_0$ finiement engendré, alors $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$ pour toute mesure de probabilité borélienne μ qui est invariante par f. Nous comparons également certains résultats obtenus pour les difféomorphismes de surfaces avec des résultats analogues pour les difféomorphismes du cercle.

1. Introduction

In his seminal article [**18**] S. Smale outlined a program for the investigation of the properties of generic smooth dynamical systems. He proposed as definition of the

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object of study the smooth action of a non-compact Lie group $\mathcal{G}$ on a manifold M; i.e., a smooth function

$$f : \mathcal{G} \times M \to M$$

satisfying $f(g_1, f(g_2, x)) = f(g_1 g_2, x)$ and $f(e, x) = x$ for all $x \in M$ and all $g_1, g_2 \in \mathcal{G}$, where e is the identity of $\mathcal{G}$. Equivalently one can consider the homomorphism

$$\phi : \mathcal{G} \to \mathrm{Diff}(M)$$

from $\mathcal{G}$ to the group of diffeomorphisms of M given by $\phi(g)(x) = f(g, x)$. The primary motivation, and by far the most studied case, has been that where $\mathcal{G}$ is either the Lie group $\mathbb{R}$ of real numbers or the discrete group $\mathbb{Z}$. As noted in the Introduction to this volume this study grew out of an interest in solution of differential equations where the group $\mathbb{R}$ or $\mathbb{Z}$ represents time (continuous or discrete).

In this article we will focus on the far less investigated case where $\mathcal{G}$ is a subgroup of a Lie group of dimension greater than one. The continuous and discrete cases when $\mathcal{G}$ is $\mathbb{R}$ or $\mathbb{Z}$ share many characteristics with each other and it is often clear how to formulate (or even prove) an analogous result in one context based on a result in the other. Very similar techniques can be used in the two contexts. However, when we move to more complicated groups the difference between the actions of a connected Lie group and the actions of a discrete subgroup become much more pronounced. One must start with new techniques in the investigation of actions of a discrete subgroup of a Lie group.

As in the case of actions by $\mathbb{R}$ and $\mathbb{Z}$ one can impose additional structures on M, such as a volume form or symplectic form, and require that the group $\mathcal{G}$ preserve them. For this article we consider manifolds of dimension two where the notion of volume form and symplectic form coincide. As it happens many of the results we will discuss are valid when a weaker structure, namely a Borel probability measure, is preserved.

The main object of this article is to provide some context for, and an exposition of, joint work of the author and Michael Handel which can be found in [**8**].

The ultimate aim is the study of the (non)-existence of actions of lattices in a large class of non-compact Lie groups on surfaces. A definitive analysis of the analogous question for actions on S^1 was carried out by É. Ghys in [**9**]. Our approach is topological and insofar as possible we try to isolate properties of a group which provide the tools necessary for our analysis. The two key properties we consider are almost simplicity of a group and the existence of a distortion element. Both are defined and described below.

We will be discussing groups of homeomorphisms and diffeomorphisms of the circle S^1 and of a compact surface S without boundary. We will denote the group of C^1 diffeomorphisms which preserve orientation by $\mathrm{Diff}(X)$ where X is S^1 or S. Orientation preserving homeomorphisms will be denoted by $\mathrm{Homeo}(X)$. If μ is a Borel probability measure on X then $\mathrm{Diff}_\mu(X)$ and $\mathrm{Homeo}_\mu(X)$ will denote the respective subgroups which preserve μ. Finally for a surface S we will denote by $\mathrm{Diff}_\mu(S)_0$ the subgroup of $\mathrm{Diff}_\mu(S)$ of elements isotopic to the identity.

An important motivating conjecture is the following.

***Conjecture 1.1* (R. Zimmer [21]).** — *Any C^∞ volume preserving action of $SL(n,\mathbb{Z})$ on a compact manifold with dimension less than n, factors through an action of a finite group.*

This conjecture suggests a kind of exceptional rigidity of actions of $SL(n,\mathbb{Z})$ on manifolds of dimension less than n. The following result of D. Witte, which is a special case of his results in [**20**], shows that in the case of $n = 3$ and actions on S^1 there is indeed a very strong rigidity.

***Theorem 1.2* (D. Witte [20]).** — *Let $\mathcal{G}$ be a finite index subgroup of $SL(n,\mathbb{Z})$ with $n \geq 3$. Any homomorphism*

$$\phi : \mathcal{G} \to \mathrm{Homeo}(S^1)$$

has a finite image.

Proof. — We first consider the case $n = 3$. If G has finite index in $SL(3,\mathbb{Z})$ then there is $k > 0$ such that

$$a_1 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a_2 = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$a_4 = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \ \text{ and } a_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix},$$

are all in $\mathcal{G}$. We will show that each of the a_i^k is in the kernel of ϕ. A result of Margulis (see Theorem 3.2 below) then implies that the kernel of ϕ has finite index. This result also implies that the case $n = 3$ is sufficient to prove the general result.

A straightforward computation shows that $[a_i, a_{i+1}] = e$ and $[a_{i-1}, a_{i+1}] = a_i^{\pm k}$, where the subscripts are taken modulo 6. Indeed $[a_{i-1}^m, a_{i+1}^n] = a_i^{\pm mnk}$.

Let $g_i = \phi(a_i)$. The group $\mathcal{H}$ generated by g_1 and g_3 is nilpotent and contains g_2^k in its center. Since nilpotent groups are amenable there is an invariant measure for the group $\mathcal{H}$ and hence the rotation number $\rho : \mathcal{H} \to \mathbb{R}/\mathbb{Z}$ is a homomorphism. Since g_2^k is a commutator, it follows that g_2^k has zero rotation number and hence it has a fixed point. A similar argument shows that for all i, g_i^k has a fixed point.

We will assume that one of the g_i^k, for definiteness say g_1^k, is not the identity and show this leads to a contradiction.

Let U_1 be any component of $S^1 \setminus \mathrm{Fix}(g_1^k)$. Then we claim that there is a $U_2 \subset S^1$ which properly contains U_1 and such that U_2 is either a component of $S^1 \setminus \mathrm{Fix}(g_6^k)$ or a component of $S^1 \setminus \mathrm{Fix}(g_2^k)$. We postpone the proof of the claim and complete the proof.

Assuming the claim suppose that U_2 is a component of $S^1 \setminus \mathrm{Fix}(g_2^k)$ the other case being similar. Then again applying the claim, this time to g_2^k we see there is U_3 which properly contains U_2 and must a component of $S^1 \setminus \mathrm{Fix}(g_3^k)$ since otherwise U_1 would properly contain itself. But repeating this we obtain proper inclusions

$$U_1 \subset U_2 \ldots U_5 \subset U_6 \subset U_1,$$

which is a contradiction. Hence $g_1^k = id$ which implies that $a_1^k \in Ker(\phi)$. A further application of the result of Margulis (Theorem 3.2 below) implies that $Ker(\phi)$ has finite index in $\mathcal{G}$ and hence that $\phi(\mathcal{G})$ is finite.

To prove the claim we note that U_1 is an interval whose endpoints are fixed by g_1^k and we will will first prove that it is impossible for these endpoints also to be fixed by g_6^k and g_2^k. This is because in this case we consider the action induced by the two homeomorphisms $\{g_6^k, g_2^k\}$ on the circle obtained by quotienting U_1 by g_1^k. These two circle homeomorphisms commute because $[g_6^k, g_2^k] = g_1^{\pm k^2}$ on $\mathbb{R}$ so passing to the quotient where g_1 acts as the identity we obtain a trivial commutator. It is an easy exercise to see that if two degree one homeomorphisms of the circle, f and g, commute then any two lifts to the universal cover must also commute. (E.g. show that $[\tilde{f}, \tilde{g}]^n$ is uniformly bounded independent of n.) But this is impossible in our case because the universal cover is just U_1 and $[g_6^k, g_2^k] = g_1^{\pm k^2} \neq id$.

To finish the proof of the claim we note that if U_1 contains a point $b \in \operatorname{Fix}(g_2^k)$ then $g_1^{nk}(b) \in \operatorname{Fix}(g_2^k)$ for all n and hence

$$\lim_{n\to\infty} g_i^{nk}(b) \text{ and } \lim_{n\to-\infty} g_i^{nk}(b),$$

which are the two endpoints of U_1 must be fixed by g_2^k. A similar argument applies to g_6^k.

It follows that at least one of g_6^k and g_2^k has no fixed points in U_1 and does not fix both endpoints. I.e. there is U_2 as claimed. □

It is natural to ask the analogous question for surfaces.

Example 1.3. — *The group $SL(3, \mathbb{Z})$ acts smoothly on S^2 by projectivizing the standard action on $\mathbb{R}^3$.*

Consider S^2 as the set of unit vectors in $\mathbb{R}^3$. If $x \in S^2$ and $g \in SL(3, \mathbb{Z})$, we can define $\phi(g) : S^2 \to S^2$ by

$$\phi(g)(x) = \frac{gx}{|gx|}.$$

Question 1.4. — *Can the group $SL(3, \mathbb{Z})$ act continuously or smoothly on a surface of genus at least one? Can the group $SL(4, \mathbb{Z})$ act continuously or smoothly on S^2?*

2. Distortion in Groups

A key concept in our analysis of groups of surface homeomorphisms is the following.

Definition 2.1. — *An element g in a finitely generated group G is called* distorted *if it has infinite order and*

$$\liminf_{n\to\infty} \frac{|g^n|}{n} = 0,$$

where $|g|$ denotes the minimal word length of g in some set of generators. If $\mathcal{G}$ is not finitely generated then g is distorted if it is distorted in some finitely generated subgroup.

It is not difficult to show that if $\mathcal{G}$ is finitely generated then the property of being a distortion element is independent of the choice of generating set.

Example 2.2. — *The subgroup G of $SL(2,\mathbb{R})$ generated by*

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfies

$$A^{-1}BA = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = B^4 \text{ and } A^{-n}BA^n = B^{4^n}$$

so B is distorted.

Example 2.3. — *The group of integer matrices of the form*

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

is called the Heisenberg group.

If

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

then their *commutator* $f = [g,h] := g^{-1}h^{-1}gh$ is

$$f = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } f \text{ commutes with } g \text{ and } h.$$

This implies

$$[g^n, h^n] = f^{n^2}$$

so f is distorted.

Let ω denote Lebesgue measure on the torus T^2..

***Example 2.4* (G. Mess [14])**. — *In the subgroup of* $\mathrm{Diff}_\omega(\mathrm{T}^2)$ *generated by the automorphism given by*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and a translation $T(x) = x + w$ where $w \neq 0$ is parallel to the unstable manifold of A, the element T is distorted.

Proof. — Let λ be the expanding eigenvalue of A. The element $h_n = A^nTA^{-n}$ satisfies $h_n(x) = x + \lambda^n w$ and $g_n = A^{-n}TA^n$ satisfies $g_n(x) = x + \lambda^{-n}w$. Hence $g_nh_n(x) = x + (\lambda^n + \lambda^{-n})w$. Since $trA^n = \lambda^n + \lambda^{-n}$ is an integer we conclude

$$T^{trA^n} = g_nh_n, \text{ so } |T^{trA^n}| \leq 4n + 2.$$

But

$$\lim_{n\to\infty} \frac{n}{trA^n} = 0,$$

so T is distorted. □