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Jörg Wildeshaus

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NOTES ON ARTIN-TATE MOTIVES

by

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Abstract. – In this paper, we study the main structural properties of the triangulated category of Artin-Tate motives over a perfect base field k. We first analyze its weight structure, building on the main results of [**7**]. We then study its t-structure, when k is algebraic over $\mathbb{Q}$, generalizing the main result of [**19**]. We finally exhibit the interaction of the weight structure and the t-structure. When k is a number field, this will give a useful criterion identifying the weight structure *via* realizations.

***Résumé* (Notes sur les motifs d'Artin-Tate).** – Dans cet article, on étudie les propriétés structurales principales de la catégorie triangulée des motifs d'Artin-Tate sur un corps parfait k. On analyse d'abord sa structure de poids, utilisant les résultats principaux de [**7**]. Puis, on étudie sa t-structure, quand k est algébrique sur $\mathbb{Q}$; ceci généralise les résultats principaux de [**19**]. Enfin, on précise l'interaction de la structure de poids et de la t-structure. Quand k est un corps de nombres, ceci donne un critère utile permettant de caractériser la structure de poids à l'aide des réalisations.

Introduction

The aim of this article is to exhibit the basic structural properties of the $\mathbb{Q}$-linear *triangulated category of Artin-Tate motives* over a fixed perfect base field k. The definition of this category will be recalled, and a number of generalizations will be defined in Section 1. Roughly speaking, the properties we shall be interested in, then fall into two classes.

First (Section 2), we apply the main results of [**7**] to Artin-Tate motives. More precisely, we start by recalling Bondarko's foundational notion of *weight structure*, and his construction of such a structure on the category of *geometrical motives* [**29**, Chapter 5]. We shall refer to it as the *motivic weight structure*. We then show (Theorem 2.5)

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that the motivic weight structure induces a weight structure on the triangulated category of Artin-Tate motives. We also give a very explicit description of the *heart* of the latter, showing in particular that it is Abelian semi-simple. This latter property requires the passage to rational coefficients; indeed, in order to simplify the exposition, we chose from the very beginning to present only the $\mathbb{Q}$-linear theory. The interested reader will note that, as suggested by the main results of [**7**], most of the results of Section 2 admit integral versions. Notable exceptions are Corollaries 2.7 and 2.9, which rely on semi-simplicity of the heart.

Second (Section 3), we generalize the main result from [**19**] from Tate motives to Artin-Tate motives, when the base field is algebraic over $\mathbb{Q}$. More precisely, we show (Theorem 3.1) that under this hypothesis, there is a non-degenerate *t-structure* on the triangulated category of Artin-Tate motives. The strategy of proof is identical to the one used by Levine. By definition, the *heart* $MAT(k)$ of this t-structure is the Abelian category of *mixed Artin-Tate motives.* Let us note that an equivalent construction of the category $MAT(k)$ is given in [**14**, Sect. 2.17]. Using the main result of [**32**], we show (Corollary 3.4) that the triangulated category of Artin-Tate motives is canonically equivalent to the bounded derived category of $MAT(k)$.

Our main interest lies then in the simultaneous application of both points of view: that of weight structures and that of t-structures. Still assuming that k is algebraic over $\mathbb{Q}$, we give a characterization (Theorem 3.9) of the weight structure on the triangulated category of Artin-Tate motives in terms of the t-structure. Specializing further to the case of number fields, we get a powerful criterion (Theorem 3.11), allowing to identify the weight structure *via* the weights occurring in the Hodge theoretic or ℓ-adic *realization.* This criterion is at the heart of our construction of the *interior motive* of Kuga-Sato families over Hilbert-Blumenthal varieties [**33**].

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Notation and conventions. – Throughout the article, k denotes a fixed perfect base field. The notation of this paper follows that of [**29**, Chapter 5], and is thus compatible with the one used in Levine's contribution to these proceedings [**20**]. Thus, the reader may consult [**29**, Chapter 5] or [**20**, Lecture 1] for the definition of the triangulated categories $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ and $DM_{\mathrm{gm}}(k)$ of (effective) geometrical motives over k. Using [**28**, Cor. 4.9] ([**29**, Thm. 4.3.1] if k admits resolution of singularities; cmp. also [**20**, Sect. 10]), we canonically identify $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ with a full triangulated subcategory of $DM_{\mathrm{gm}}(k)$. Let F be a commutative $\mathbb{Q}$-algebra. The notation $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)_F$ and $DM_{\mathrm{gm}}(k)_F$ stands for the F-linear analogues of these triangulated categories defined in [**1**, Sect. 16.2.4 and Sect. 17.1.3]. Similarly, let us denote by $CHM^{\mathrm{eff}}(k)$ and $CHM(k)$ the categories opposite to the categories of (effective) Chow motives, and by $CHM^{\mathrm{eff}}(k)_F$ and $CHM(k)_F$ the pseudo-Abelian completion of the category

$CHM^{\text{eff}}(k) \otimes_{\mathbb{Z}} F$ and $CHM(k) \otimes_{\mathbb{Z}} F$, respectively. Let us connect this notation to the one introduced in Sujatha's contribution to these proceedings [**26**]: we have

$$CHM^{\text{eff}}(k) = \mathbf{Mot}^{\text{eff}}_{\text{rat}}(k,\mathbb{Z})^{\text{opp}} \quad \text{and} \quad CHM(k) = \mathbf{Mot}_{\text{rat}}(k,\mathbb{Z})^{\text{opp}} \ ;$$

$$CHM^{\text{eff}}(k)_F = \mathbf{Mot}^{\text{eff}}_{\text{rat}}(k,F)^{\text{opp}} \quad \text{and} \quad CHM(k)_F = \mathbf{Mot}_{\text{rat}}(k,F)^{\text{opp}}$$

[**26**, Sect. 3]. As recalled in [loc. cit.], the functor M associating to a smooth, projective variety X over k its Chow motive $M(X) \in CHM^{\text{eff}}(k)$ is therefore covariant. Using [**27**, Cor. 2] ([**29**, Cor. 4.2.6] if k admits resolution of singularities; cmp. also [**20**, Cor. 10.2]), we canonically identify $CHM^{\text{eff}}(k)_F$ and $CHM(k)_F$ with a full additive sub-category of $DM^{\text{eff}}_{\text{gm}}(k)_F$ and $DM_{\text{gm}}(k)_F$, respectively.

1. Definition and first properties

Fix a commutative $\mathbb{Q}$-algebra F, which we suppose to be semi-simple and Noetherian, in other words, a finite direct product of fields of characteristic zero. In this section, we recall the definition of the F-linear triangulated category of Artin-Tate motives (Definition 1.3), and define a number of variants, indexed by certain semi-simple sub-categories of the category of discrete representations of the absolute Galois group of our perfect base field k (Definition 1.6). We then start the analysis of these categories, following the part of [**19**] valid without additional assumptions on k.

For any integer m, there is defined a Tate object $\mathbb{Z}(m)$ in $DM_{\text{gm}}(k)$, which belongs to $DM^{\text{eff}}_{\text{gm}}(k)$ if $m \geq 0$ [**29**, p. 192]. We shall use the same notation when we consider $\mathbb{Z}(m)$ as an object of $DM_{\text{gm}}(k)_F$.

Definition 1.1 **(cmp. [19, Def. 3.1]).** – Define the *triangulated category of Tate motives over* k as the strict full triangulated sub-category $DMT(k)_F$ of $DM_{\text{gm}}(k)_F$ generated by the $\mathbb{Z}(m)$, for $m \in \mathbb{Z}$.

Recall that by definition, a strict sub-category is closed under isomorphisms in the ambient category. Note that the family $\{\mathbb{Z}(m), m \in \mathbb{Z}\}$ is closed under tensor product. We claim that this property implies that the category $DMT(k)_F$ is tensor triangulated. Indeed, we may define a filtration by strict full sub-categories

$$DMT(k)^0_F \subset DMT(k)^1_F \subset \cdots \subset DMT(k)_F$$

as follows: objects in $DMT(k)^0_F$ are finite direct sums of objects isomorphic to $\mathbb{Z}(m)[r]$, for $m, r \in \mathbb{Z}$. Then define $DMT(k)^n_F$, $n \geq 1$ inductively as the category of cones of morphisms

$$M_1 \longrightarrow M_2, \ \text{for } M_1, M_2 \in DMT(k)^{n-1}_F \,.$$

Since the generating family is closed under tensor product, so is $DMT(k)^0_F$. By induction, the tensor product of an object of $DMT(k)^{n_1}_F$ and an object of $DMT(k)^{n_2}_F$ belongs to $DMT(k)^{n_1+n_2}_F$. Altogether, the category

$$DMT(k)_F = \bigcup_{n \in \mathbb{Z}} DMT(k)^n_F$$

is tensor triangulated, as claimed.

Let us compare our notation with the one used in Levine's contribution to these proceedings [**20**]: when the coefficients F equal the field of rational numbers, then $DMT(k)_F$ equals the category denoted $DMT(k)$ in [**20**, Sect. 12].

Definition 1.2. – Define the *triangulated category of Artin motives over* k as the strict full triangulated sub-category $DMA(k)_F$ of $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)_F$ generated by direct factors of the motives $M(X)$ of smooth, zero-dimensional schemes X over k.

One sees as above that this category is again tensor triangulated.

Definition 1.3. – Define the *triangulated category of Artin-Tate motives over* k as the strict full tensor triangulated sub-category $DMAT(k)_F$ of $DM_{\mathrm{gm}}(k)_F$ generated by $DMA(k)_F$ and $DMT(k)_F$.

Thus, $DMAT(k)_F$ is the strict full triangulated sub-category of $DM_{\mathrm{gm}}(k)_F$ generated by the objects $M \otimes \mathbb{Z}(m)$, for $M \in DMA(k)_F$ and $m \in \mathbb{Z}$. The following observation [**29**, Remark 2 on p. 217] is vital.

Proposition 1.4. – *The triangulated category $DMA(k)_F$ of Artin motives is canonically equivalent to $D^b(MA(k)_F)$, the bounded derived category of the Abelian category $MA(k)_F$ of discrete representations of the absolute Galois group of k in finitely generated F-modules.*

More precisely, if X is smooth and zero-dimensional over k, and $\bar{k}$ a fixed algebraic closure of k, then the absolute Galois group of k, when identified with the group of automorphisms of $\bar{k}$ over k, acts canonically on the set of $\bar{k}$-valued points of X. The object of $MA(k)_F$ corresponding to $M(X)$ under the equivalence of Proposition 1.4 is nothing but the formal F-linear envelope of this set, with the induced action of the Galois group. Note that the category $MA(k)_F$ is semi-simple.

Corollary 1.5. – *There is a canonical non-degenerate t-structure on the category $DMA(k)_F$. Its heart is equivalent to $MA(k)_F$.*

Recall that a *t-structure* on a triangulated category $\mathscr{C}$ is a pair $(\mathscr{C}^{t\leq 0}, \mathscr{C}^{t\geq 0})$ of strict full sub-categories of $\mathscr{C}$, such that, putting

$$\mathscr{C}^{t\leq n} := \mathscr{C}^{t\leq 0}[-n], \quad \mathscr{C}^{t\geq n} := \mathscr{C}^{t\geq 0}[-n] \quad \forall\, n \in \mathbb{Z},$$

the following conditions are satisfied [**4**, Déf. 1.3.1].

(A) We have the inclusions

$$\mathscr{C}^{t\leq 0} \subset \mathscr{C}^{t\leq 1}, \quad \mathscr{C}^{t\geq 0} \supset \mathscr{C}^{t\geq 1}$$

of full sub-categories of $\mathscr{C}$.

(B) For any pair of objects $M \in \mathscr{C}^{t\leq 0}$ and $N \in \mathscr{C}^{t\geq 1}$, we have

$$\mathrm{Hom}_{\mathscr{C}}(M, N) = 0.$$