Motives and automorphic representations

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Abstract. – This is a survey of the conjectures, and known facts, about the relation between the Grothendieck motives of varieties over number fields, and automorphic forms.

***Résumé* (Motifs et représentations automorphes).** – Ce texte est un exposé des relations connues et conjecturées entre motifs des variétés algébriques sur les corps de nombres (au sens de Grothendieck) et représentations ou formes automorphes.

Introduction

These are, slightly expanded, the notes of my 3-hour lecture at IHÉS in July 2006.

The organizers had assigned me two tasks. First, to give an overview of the conjectural relations between the motives defined by algebraic varieties over number fields and automorphic forms—here, automorphic forms on $GL(n)$. (The relation is mediated by two other kinds of objects, namely, Galois representations and L-functions). Secondly, to furnish—in a more leisurely way, at least for the topics I chose to present, than in his paper—the prerequisites necessary to understand the automorphic part of R. Taylor's beautiful lecture, "Galois representations", at the Beijing ICM (2002)(1).

These are rather succinct notes, so it was certainly not possible to present in detail, with proofs, the basic material: the reader will have to consult the standard references [**39, 48, 18**]. My purpose was rather to introduce a listener, not necessarily familiar with the modern theory of automorphic forms, to the basic objects of the theory—at least those pertinent in relation with the theory of motives, and the standard "dictionary" relating automorphic representations and motives.

The automorphic data, and their properties, are presented in Lecture 1; the dictionary in Lecture 2.

Key words and phrases. – Automorphic forms, Langlands L-functions, Grothendieck motives.

(1) The reader will consult, not the ICM text, but the full version published in Ann. Fac. Sc. Toulouse, and available on his web site.

In both I have tried, not only to summarize *results*, but to state *problems* which may catch the fancy of young mathematicians. This is already true in Lecture 1—see (1.12), (1.14), the end of § 1.2, as well as (1.19). In Lecture 2, I have seized the opportunity given by these lectures to discuss in some detail the relation between classical and automorphic objects (§ 2.3), answering some questions often asked by arithmetic geometers reading **[18]**. § 2.4-2.6 discuss some interesting problems which may be considered using both sides of the dictionary: the existence of motives with small ramification, and the degeneracy of totally even geometric Galois representations (over $\mathbb{Q}$). Consequently these notes are rather lopsided—Lecture 2 is heavier—but I hope that the new material will be found interesting.

My course of lectures took place between those by L. Fargues (on the theory of complex multiplication) and M. Harris (on the proof of the Sato-Tate Conjecture). Fargues' lectures dealt with the motives that are "potentially Abelian" and the associated automorphic objects and L-functions. This may be seen as a special case but the theory is, there, essentially complete. On the other hand, my lectures were also intended as introducing part of the material needed by Harris. In particular, Lecture 3 introduces the Galois representations associated to self-dual automorphic representations, which play a crucial role in the proof of Sato-Tate.

In conclusion I want to thank the organizers, and particularly J.-B. Bost and J.-M. Fontaine, for a very exciting conference; the audience for their sustained interest; and J-P. Serre, M. Harris, J.-M. Fontaine and N. Ratazzi for useful discussions. I also thank Bost and Fontaine for allowing me to include here a long-overdue Errata to **[18]**.

Addendum (July 2011). – Given the long delay in publication, there has been impressive progress in the field since these notes were written. It was not possible to rewrite them in order to include the new developments. I will simply refer to the recent papers, most of them in preprint form.

- The construction of (compatible systems of) Galois representations associated to suitable cuspidal representations for totally real or CM-fields is now complete. See **[53]** as well as the volume announced at the end of our Chapter 3, **[16]**.
- The existence of cuspidal representations associated to compatible systems of Galois representations has now been proved in considerable generality. (One proves it only "potentially," i.e., one obtains the sought representation of $GL(n)$ only for a suitable Galois extension.) For the most general results to date see **[5]**. This relies on important work done in the meantime, in particular by Harris and the authors of this paper.
- In particular the Sato-Tate conjecture is now known with no ramification condition **[6]**.
- The Ramanujan conjecture (Theorem 3.10) is now known under the natural assumptions (F totally real or CM, π cohomological and essentially self-dual.) See **[53]**, **[12]**. For a simple proof at the unramified primes see **[20]**.

Addendum (June 2016). – As the reader will notice, there has occurred a further delay in publication. Consequently, some parts of the text are out of date. See the Foreword to this volume for the main new contributions. In particular, the reader should be aware that the self-duality condition which appears in these notes is no longer required for the existence of Galois representations. Moreover the relations between forms on $GL(n)$ and classical groups have been proved. See [**3, 45**].

Lecture 1: Algebraic representations

1.1. In this lecture F is a number field (the *ground field*),

$$G = GL(n).$$

If F is given I simply note

$$\mathbb{A} = \mathbb{A}_F = \text{adèles of } F.$$

I use standard notations: thus $\mathbb{A}_F = \prod_v' F_v$ (restricted product) where v denotes a prime (finite or Archimedean) of F, F_v is the completion at v; $\mathcal{O}_v$ is the ring of integers for v finite, ϖ_v a uniformizing parameter.

I want to define in full generality the *automorphic* objects that should eventually be identified with motives.

Fix ω = continuous character of $F^\times \backslash \mathbb{A}^\times$—*not necessarily unitary.*

We will be interested in automorphic forms for the group G, i.e., functions on $G(F)\backslash G(\mathbb{A})$ (verifying certain conditions).

We identify Z = center of G with $\mathbb{G}_m$, $Z(\mathbb{A}) = \mathbb{A}^\times$, and set :

Definition. – *We define*

$$\mathcal{A}_G^{\mathrm{cusp}}(\omega) = \{f : G(F)\backslash G(\mathbb{A}) \to \mathbb{C},\ f(zg) \equiv \omega(z)\, f(g) \quad (z \in Z(\mathbb{A})),\ f \ \textit{cuspidal}\},$$

with the right representation of $G(\mathbb{A})$.

Recall that "cuspidal" functions = cusp forms are, in this context, defined by the condition

$$\int_{N(F)\backslash N(\mathbb{A})} f(n\,g)\,dn \equiv 0 \qquad \text{(identically in } g\text{)},$$

$$N = \text{unipotent radical of a proper parabolic subgroup of } G.$$

The last condition means simply that, in block-matrix form,

$$N = \begin{pmatrix} 1 & * & * & \dots & * \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & \ddots & * \\ & & & & 1 \end{pmatrix}$$
$$\underbrace{\quad}_{n_1}\ \underbrace{\quad}_{n_2}\ \cdots\ \underbrace{\quad}_{n_r}$$

with $n = \Sigma\, n_i$, $(n_i) \neq n$.

There is also a mild "L^2" condition which is a little complicated to define, because ω is not unitary: use that

$$G(\mathbb{A}) = A_G\ G(\mathbb{A})^1$$

where $A_G = \mathbb{R}_+^\times \subset \mathbb{A}^\times$ (diagonal embedding),

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) : |\det g| = 1\},$$

and that $G(F)\backslash G(\mathbb{A})^1$ has finite volume for the invariant measure.

With these definitions,

$$\mathscr{A}_G^{\mathrm{cusp}}(\omega) = \widehat{\bigoplus}\ \pi \tag{1.1}$$

(countable direct sum of irreducible representations of $G(\mathbb{A})$ in Hilbert spaces) and by definition, each summand

$$\pi = \text{cuspidal representation of } G(\mathbb{A}), \text{ with central character } \omega.$$

Remark. – $G(\mathbb{A}_f) = \prod\limits_{v\ \mathrm{finite}}' G(F_v)$ (restricted product) has compact-open subgroups, of the form

$$K = \prod_{\substack{\text{finite set}\\ S \text{ of } v}} K_v \times \prod_{\text{all other } v} G(\mathscr{O}_v) \tag{1.2}$$

with $K_v \subset G(F_v)$ compact-open ($v \in S$). Classically an automorphic form f is as before, but we impose moreover:

- For some (or all) K, f is K-finite under right translation
- At the Archimedean primes, f is K_v-finite where K_v ($= O(n)$ or $U(n)$) is maximal compact in $G(F_v)$ ($= GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$).

If f belongs to only one π in (1.1), f is then $\mathfrak{Z}$-finite for the action of the center $\mathfrak{Z}$ of the enveloping algebra. I will not dwell on such matters, cf. **[9]**.

If we consider the full decomposition

$$G(\mathbb{A}) = \prod_{\text{all } v}{}' G(F_v)$$