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PURE MOTIVES, MIXED MOTIVES AND EXTENSIONS OF MOTIVES ASSOCIATED TO SINGULAR SURFACES

by

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Abstract. – We first recall the construction of the Chow motive modelling intersection cohomology of a proper surface $\overline{X}$, and study its fundamental properties. Using Voevodsky's category of effective geometrical motives, we then study the motive of the exceptional divisor D in a non-singular blow-up of $\overline{X}$. If all geometric irreducible components of D are of genus zero, then Voevodsky's formalism allows us to construct certain one-extensions of motives, as canonical sub-quotients of the motive with compact support of the smooth part of $\overline{X}$. Specializing to Hilbert-Blumenthal surfaces, we recover a motivic interpretation of a recent construction of A. Caspar.

Résumé **(Motifs purs, motifs mixtes et extensions de motifs associés aux surfaces singulières)**
On rappelle d'abord la construction du motif de Chow sous-jacent à la cohomologie d'intersection d'une surface propre $\overline{X}$, and l'on en étudie les propriétés fondamentales. En utilisant le langage des motifs effectifs géométriques à la Voevodsky, on étudie ensuite le motif du diviseur exceptionnel D dans un éclatement non-singulier de $\overline{X}$. Si toutes les composantes géométriques de D sont de genre zéro, alors le formalisme de Voevodsky permet la construction de certaines extensions de motifs, comme sous-quotients canoniques du motif à support compact de la partie lisse de $\overline{X}$. Dans le cas des surfaces de Hilbert-Blumenthal, ceci donne une interprétation motivique d'une construction récente due à A. Caspar.

Introduction

The modest aim of this article is to construct non-trivial extensions in Voe'vodsky's category of effective geometrical motives, by studying a very special and concrete geometric situation, namely that of a singular proper surface.

This example illustrates a much more general principle: varieties Y that are singular (or non-proper, for that matter), can provide interesting extensions of motives. The cohomological theories of mixed sheaves suggest where to look for these motives: the

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one should come from the open smooth part Y_{reg} of Y—the *intersection motive* of Y—the other should be constructed out of the complement of Y_{reg} in (a compactification of) Y—the *boundary motive* of Y_{reg}. This principle (for which no originality is claimed, since it has been part of the mathematical culture for some time) will be discussed in more detail separately [**23**], in order to preserve the structure of the present article. It is intended as a research article with a large instructional component.

The geometric object of interest is a proper surface $\overline{X}$ over an arbitrary base field k.

The first three sections contain nothing fundamentally new, except maybe for the systematic use of Künneth filtrations (which are canonical) instead of Künneth decompositions (which in general are not). Section 1 reviews a special case of a result of Borho and MacPherson [**4**], computing the intersection cohomology of $\overline{X}$ in terms of the cohomology of a desingularization $\widetilde{X}$. The result, predicted by the Decomposition Theorem of [**3**], implies that the former is a direct factor of the latter. More precisely (Theorem 1.1), its complement is given by the second cohomology of the exceptional divisor D of $\widetilde{X}$. As remarked already by de Cataldo and Migliorini [**6**], this fact can be interpreted motivically, which allows one to construct the intersection motive $h_{!*}(\overline{X})$ of $\overline{X}$. This is done in Section 2. We get a canonical decomposition

$$h(\widetilde{X}) = h_{!*}(\overline{X}) \oplus \bigoplus_m h^2(D_m)$$

in the category of Chow motives over k. Recall that this category is pseudo-Abelian. The above decomposition should be considered as remarkable: to construct a sub-motive of $h(\widetilde{X})$ does not *a priori* necessitate the *identification*, but only the *existence* of a complement. In our situation, the complement *is* canonical, thanks to the very special geometrical situation. This point is reflected by the rather subtle functoriality properties of $h_{!*}(\overline{X})$ (Proposition 2.5): viewed as a sub-motive of $h(\widetilde{X})$, it is respected by pull-backs, viewed as a quotient, it is respected by push-forwards under dominant morphisms of surfaces. Section 3 is devoted to the existence and the study of the Künneth filtration of $h_{!*}(\overline{X})$. The main ingredient is of course Murre's construction of Künneth projectors for the motive $h(\widetilde{X})$ [**14**]. Theorem 3.8 shows how to adapt these to our construction.

As suggested by one of the fundamental properties of intersection cohomology [**3**], the intersection motive of $\overline{X}$ satisfies the Hard Lefschetz Theorem for ample line bundles on $\overline{X}$. We prove this result (Theorem 4.1) in Section 4. In fact, we give a slight generalization (Variant 4.2), which will turn out to be useful for the setting we shall study in the last section.

Section 5 is concerned with the motive of the boundary D of the desingularization $\widetilde{X}$ of $\overline{X}$. This boundary being singular in general, the right language for the study of its motive is given by Voevodsky's triangulated category of effective geometrical motives [**21**, Chapter 5]. The section starts with a review of the definition of this category, and of its relation to Chow motives. It is then easy to define motivic analogues of H^0 and H^2 of D, and to see that they are Chow motives. The most interesting part is the motivic analogue of the part of degree one H^1, which will be seen as a canonical sub-quotient of the motive of D.

In Section 6, we unite what was said before, and give our main result (Theorem 6.6). Assuming that all geometric irreducible components of D are of genus zero, we construct a one-extension of the degree two-part of the intersection motive of $\overline{X}$ by the degree one-part of the motive of D. We have no difficulty to admit that this statement was greatly inspired by the main result of a recent article of Caspar [**5**]. It thus appeared appropriate to conclude this article by a discussion of his result. This is done in Section 7, where we show that in the geometric setting considered in [loc. cit.], Theorem 6.6 yields a motivic interpretation of Caspar's construction.

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Notations and convention. – k denotes a fixed base field, and CH stands for the tensor product with $\mathbb{Q}$ of the Chow group. The $\mathbb{Q}$-linear category of Chow motives over k is denoted by $\mathrm{CHM}(k)_{\mathbb{Q}}$. Our standard reference for Chow motives is Scholl's survey article [**18**].

1. Intersection cohomology of surfaces

In order to motivate the construction of the intersection motive, to be given in the next section, we shall recall the computation of the *intersection cohomology* of a complex surface.

Thus, throughout this section, our base field k will be equal to $\mathbb{C}$. We consider the following situation:

$$X \overset{j}{\hookrightarrow} X^* \overset{i}{\hookleftarrow} Z.$$

The morphism i is a closed immersion of a sub-scheme Z, with complement j. The scheme X^* is a surface over $\mathbb{C}$, all of whose singularities are contained in Z. Thus, the surface X is smooth.

Our aim is to identify the intersection cohomology groups $H^n_{!*}(X^*(\mathbb{C}),\mathbb{Q})$. Note that since X is smooth, the complex $\mathbb{Q}_X[2]$ consisting of the constant local system $\mathbb{Q}$, placed in degree -2, can be viewed as a *perverse sheaf* (for the middle perversity) on $X(\mathbb{C})$ [**3**, Sect. 2.2.1]. Hence its *intermediate extension* $j_{!*}\mathbb{Q}_X[2]$ [**3**, (2.2.3.1)] is defined as a perverse sheaf on $X^*(\mathbb{C})$. By definition,

$$H^n_{!*}(X^*(\mathbb{C}),\mathbb{Q}) = H^{n-2}(X^*(\mathbb{C}), j_{!*}\mathbb{Q}_X[2]),\ \forall\, n \in \mathbb{Z}.$$

In order to identify $H^n_{!*}(X^*(\mathbb{C}),\mathbb{Q})$, note first that the normalization of X^* is finite over X^*, and the direct image under finite morphisms is exact for the perverse t-structure [**3**, Cor. 2.2.6 (i)]. Therefore, intersection cohomology is invariant under passage to the normalization. In the sequel, we therefore assume X^* to be normal. In particular, its singularities are isolated.

Next, note that if X^* is smooth, then the complex $j_{!*}\mathbb{Q}_X[2]$ equals $\mathbb{Q}_{X^*}[2]$. Transitivity of $j_{!*}$ [**3**, (2.1.7.1)] shows that we may enlarge X, and hence assume that the closed sub-scheme Z is finite.

Choose a resolution of singularities. More precisely, consider in addition the following diagram, assumed to be cartesian:

$$\begin{array}{ccccc} X & \overset{\tilde{j}}{\hookrightarrow} & \widetilde{X} & \overset{\tilde{\imath}}{\hookleftarrow} & D \\ \| & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \underset{j}{\hookrightarrow} & X^* & \underset{i}{\hookleftarrow} & Z. \end{array}$$

The morphism π is assumed proper (and birational) and the surface $\widetilde{X}$, smooth. We then have the following special case of [**4**, Thm. 1.7].

Theorem 1.1. – (i) *For $n \neq 2$,*

$$H^n_{!*}(X^*(\mathbb{C}), \mathbb{Q}) = H^n(\widetilde{X}(\mathbb{C}), \mathbb{Q}).$$

(ii) *The group $H^2_{!*}(X^*(\mathbb{C}), \mathbb{Q})$ is a direct factor of $H^2(\widetilde{X}(\mathbb{C}), \mathbb{Q})$, with a* canonical *complement. As a sub-group, this complement is given by the map*

$$\tilde{\imath}_* : H^2_{D(\mathbb{C})}(\widetilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(\widetilde{X}(\mathbb{C}), \mathbb{Q})$$

from cohomology with support in $D(\mathbb{C})$; this map is injective. As a quotient, the complement is given by the restriction

$$\tilde{\imath}^* : H^2(\widetilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(D(\mathbb{C}), \mathbb{Q}) \ ;$$

this map is surjective.

Note that this result is compatible with further blow-up of $\widetilde{X}$ in points belonging to D.

Let us construct the maps between $H^n_{!*}(X^*(\mathbb{C}), \mathbb{Q})$ and $H^n(\widetilde{X}(\mathbb{C}), \mathbb{Q})$ leading to the above identifications. Consider the total direct image $\pi_*\mathbb{Q}_{\widetilde{X}}$; following the convention used in [**3**], we drop the letter "R" from our notation.

Lemma 1.2. – *The complex $\pi_*\mathbb{Q}_{\widetilde{X}}[2]$ is a perverse sheaf on X^*.*

Proof. – Let P be a point (of Z) over which π is not an isomorphism, and denote by i_P its inclusion into X^*. By definition [**3**, Déf. 2.1.2], we need to check that (a) the higher inverse images $H^n i_P^* \pi_* \mathbb{Q}_{\widetilde{X}}$ vanish for $n > 2$, (b) the higher exceptional inverse images $H^n i_P^! \pi_* \mathbb{Q}_{\widetilde{X}}$ vanish for $n < 2$.

(a) By proper base change, the group in question equals $H^n(\pi^{-1}(P), \mathbb{Q})$. Since $\pi^{-1}(P)$ is of dimension at most one, there is no cohomology above degree two.

(b) The surface $\widetilde{X}$ is smooth. Duality and proper base change imply that the group in question is abstractly isomorphic to the dual of $H^{4-n}(\pi^{-1}(P), \mathbb{Q})$. This group vanishes if $4 - n$ is strictly larger than two. □