Lecture Notes on the Navier-Stokes-Fourier system: weak solutions, relative entropy inequality, weak strong uniqueness

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LECTURE NOTES ON THE NAVIER-STOKES-FOURIER SYSTEM: WEAK SOLUTIONS, RELATIVE ENTROPY INEQUALITY, WEAK STRONG UNIQUENESS

by

Antonin Novotný

Dedicated to the memory of Alexander Kazhikov.

Abstract. – We investigate three issues in the theory of the Navier-Stokes-Fourier system describing the motion of a compressible, viscous and heat conducting fluid: weak compactness of the family of weak solutions (that is the main ingredient in the proof of the existence of weak solutions), relative entropy inequality and weak-strong uniqueness principle.

1. Introduction

These Lecture Notes are devoted to some aspects of the theory of the Navier-Stokes-Fourier system. We shall discuss 1) existence of weak solutions, 2) existence of suitable weak solutions and relative entropies, 3) weak strong uniqueness property in the class of weak solutions. For physical reasons, we shall limit ourselves to the three dimensional physical space, and for the sake of simplicity, to the flows in bounded domains with no-slip boundary conditions.

1.1. Nonexhausting bibliographic remarks

1.1.1. *Weak solutions.* – There are several ways to define weak solutions for the complete Navier-Stokes-Fourier system. Here, we shall mention three of them: the convenience of each definition depends on the mathematical assumptions that one imposes on the constitutive laws for pressure (internal energy) on one hand, and on the transport coefficients on the other hand. First one and the third one are continuations of the theories based on the so called effective viscous flux identity started by P.L. Lions [**24**] and the second one, due to Bresch, Desjardins [**2**] can be considered as a continuation of theories based on new à priory estimates in the line started by Kazhikov [**38**].

The first approach due to Feireisl [**12**] based on a weak formulation of the continuity, momentum and internal energy equations is convenient for the pressure laws

$$p(\varrho,\vartheta) = p_c(\varrho) + \vartheta p_\vartheta(\varrho),$$

where

$$p_c(\varrho) \approx \varrho^\gamma,\ p_\vartheta(\varrho) \approx \varrho^\Gamma,\ \Gamma \le \gamma/3.$$

Here, ϑ denotes the temperature, ϱ the density and γ is the adiabatic coefficient of the fluid. The heat conductivity in this approach has to be temperature dependent (with a convenient polynomial growth), and the viscosity coefficients have to be constant.

The second approach due to Bresch, Desjardins [**2**], [**3**] (see also Mellet, Vasseur [**28**]) is convenient in the case when the shear viscosity μ and the bulk viscosity η depend on the density and satisfy the differential identity

$$(\eta - \frac{2}{3}\mu)'(\varrho) = 2\varrho\mu'(\varrho) - 2\mu(\varrho),$$

and the pressure satisfies

$$p(\varrho,\vartheta) = \varrho\vartheta + p_c(\varrho),$$

where $p_c(\varrho)$ is singular at $\varrho \to 0$ and behaves as ϱ^γ at $\varrho \to \infty$. The main ingredient in the proof in this situation is the fact that the particular relation between viscosities stated above makes possible to establish a new mathematical entropy identity, that provides estimates for the gradient of density. This estimate implies compactness of the sequence of approximating densities.

The third approach is due to [**14**]. It is based on the weak formulation of continuity and momentum equations and on the formulation of the conservation of energy in terms of the specific entropy that involves explicitly the second law of thermodynamics with entropy production rate being a non negative measure. This approach is applicable for the pressure laws $p(\varrho,\vartheta)$ exhibiting the coercivity of type ϱ^γ and ϑ^4 for large densities and temperatures. The viscosity coefficients are temperature dependent and have to behave as $1+\vartheta^\alpha$ $(1 \ge \alpha > 2/5)$, and the heat conductivity has to behave as $1+\vartheta^3$. This setting includes at least one physically reasonable case of a monoatomic gas in the situation when the radiation is not neglected and is given by the Stephan-Boltzman law.

The above formulation is sufficiently weak to allow existence of variational solutions for large data. On the other hand it is sufficiently robust to yield most of low Mach and low Reynolds number limits of mathematical fluid mechanics, and - more surprisingly - it obeys the weak-strong uniqueness principle.

In this Lecture Notes, we shall concentrate on the third approach. We shall discuss within this concept three issues: 1) existence of weak solutions (or, more precisely, in order to concentrate on the essence of the problem, the compactness of the family of weak solutions); 2) Relative entropy inequality for the complete system; 3) Weak strong uniqueness property in the class of weak solutions. This text does not contain any new results: it is a compilation of our recent works [**14**, Chapter3] and [**15**].

1.1.2. *Lions' approach and Feireisl's approach.* – The concept of weak solutions in fluid dynamics was introduced in 1934 by Leray [**22**] in the context of incompressible newtonian fluids. It has been extended more than 60 years later to the Newtonian compressible fluids in barotropic regime (meaning that $p = p(\varrho) \approx \varrho^\gamma$) by Lions [**24**].

The Lions' theory relies on two crucial observations:

1. A discovery of a certain weak continuity property of the quantity
$$p(\varrho) + (\frac{4}{3}\mu + \eta)\mathrm{div}\mathbf{u}$$
called effective viscous flux. This part is essential for the existence proof; it employs certain cancelation properties that are available due to the structure of the equations, that are mathematically expressed through a commutator involving density, momentum and the Riesz operator.
2. Theory of renormalized solutions to the transport equation that P.L. Lions introduced together with DiPerna in [**7**]. In the context of compressible Navier-Stokes equations, the DiPerna-Lions transport theory applies to the continuity equation. The theory asserts among others that the limiting density is a renormalized solution to the continuity equation provided it is squared integrable. This hypothesis is satisfied only provided $\gamma \geq 9/5$. The condition on the squared integrability of the density is the principal obstacle to the improvement of the Lions result.

Notice that some indications on the particular importance of the effective viscous flux have been known at about the same time to several authors and used in different problems dealing with small data (see Hoff [**20**], Padula [**29**]) and that the suggestion to use the continuity equations to evaluate the oscillations in the sequence of approximating densities has been formulated and performed in the one dimensional case by D. Serre [**34**].

All physically reasonable adiabatic coefficients γ belong to the interval $(1, 5/3]$, the value $\gamma = 5/3$ being reserved for the monoatomic gas. This is the reason why it is interesting and important to relax the condition on the adiabatic coefficient in the Lions theory. This has been done Feireisl et al. in [**16**]. The new additional aspects of this extension are based on the previous observations by Feireisl in [**11**] and are the following:

1. As suggested in [**11**], the authors have used the oscillations defect measure to evaluate the oscillations in the sequence of approximating densities, and proved that it is bounded provided $\gamma > 3/2$.
2. The boundedness of the oscillations defect measure is a criterion that replaces the condition of the squared integrability of the density in the DiPerna-Lions transport theory. Consequently if any term of the sequence of approximating densities satisfies the renormalized continuity equation, and if the oscillations defect measure of this sequence is bounded, then the weak limit of the sequence is again a renormalized solution of the continuity equation.

1.1.3. *Weak solutions for the complete Navier-Stokes-Fourier system.* – The existence theory for the complete Navier-Stokes-Fourier system (with temperature dependent viscosities) employs both Lions' and Feireisl's techniques. In addition, it presents the following difficulties:

1. In order to reduce the investigation to a situation similar to the barotropic case, one has to prove first the strong convergence of the temperature sequence. This point involves the treatment of the entropy production rate as a Radon measure and a convenient use of the compensated compactness, namely of the Div-curl lemma in combination with the theory of parametrized Young measures.
2. Even after the strong convergence of temperature is known, the weak continuity of the effective viscous flux is not an obvious issue. It requires to use another cancelation property that mathematically expresses through another commutator including shear viscosity, symmetric velocity gradient and the Riesz operator.
3. Once the weak continuity property of the effective viscous flux is known, the proof follows the lines of Lion's and Feireisl's approaches: a) one proves first the boundedness of the oscillation defect measure for the sequence of densities; b) the boundedness of oscillations defect measure implies that the limiting density is a renormalized solution to the continuity equation; c) the renormalized continuity equation is used to show that the oscillations in the density sequence do not increase in time. This means the strong convergence of density.

1.1.4. *Relative entropies and weak strong uniqueness.* – Weak solutions are not known to be uniquely determined (cf. e.g., exposition of Fefferman [**10**] dealing with three dimensional incompressible Navier-Stokes equations) and may exhibit rather pathological properties, see e. g. Hoff and Serre [**21**]. So far, the best property that one may expect in the direction of a uniqueness result, is the weak-strong uniqueness, meaning that any weak solution coincides with the strong solution emanating from the same initial data, as long as the latter exists. The weak-strong uniqueness principle is known for the incompressible Navier-Stokes equations since the celebrated paper of Leray [**22**] extended thirty years later in the well-known works of Prodi[**32**], Serrin [**35**], 1959, 1962. Since these works the subject was revisited in many papers, with so far the last word being said in [**8**].

About 50 years after Prodi and Serrin, the weak-strong uniqueness problem has been revisited by Desjardins [**6**] and Germain [**18**] for the compressible Navier-Stokes equations. They obtained some partial and conditional results. Finally, the unconditional weak strong uniqueness principle has been proved in [**13**] (see also related paper [**17**]).

Only very recently the weak strong uniqueness property has been proved in [**15**] for weak solutions of the complete Navier-Stokes-Fourier system in the entropy formulation introduced in [**14**].

In all cases cited above, the weak strong uniqueness principle has been achieved by the method of relative entropies. Relative entropy is a functional whose role is to measure the distance between a weak solution of the investigated equations and any