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NEAR-EQUILIBRIUM WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS FOR COMPRESSIBLE FLOWS

by

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Abstract. – In these notes we review the theory of weak solutions of the Navier-Stokes equations for compressible flows in near equilibrium flow regime. The study of solutions of this type was initiated by D. Hoff **[11]** with the motivation of studying the dynamics of an interface of discontinuity of solutions.

We demonstrate three results. The first is the existence of weak solutions with uniformly bounded density for the typical initial-boundary value problems. The second result improves the regularity of such weak solutions in the case when the density is piecewise Hölder continuous with a jump discontinuity across a $C^{1+\alpha}$ hypersurface. We show that the flow generated by the velocity u transports the discontinuity surface and preserve its regularity.

In the last result, we consider the fluid flows in which the density, at time $t = 0$, is piecewise smooth and the interface of discontinuity is a surface with a corner-type singularity. We show that solutions in Hoff's regularity class are well suited for the analysis of this problem and describe the phenomenon of an instantaneous change of the geometry of the interface at the point of the singularity.

1. Properties of the solutions at the interface of discontinuity

1.1. Equations. – The Navier-Stokes equations for isentropic flows express the conservation of mass and momentum:

$$\text{(N.-S.)} \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u^j) + \operatorname{div}(\rho u^j u) = \mu \Delta u^j + \lambda \operatorname{div} u_{x_j} - P(\rho)_{x_j}, \ j = 1..n, \end{cases}$$

where $(x,t) \in \Omega \times \mathbb{R}_+$, Ω is an open subset of $\mathbb{R}^n$, $n = 2,3$; ρ and $u = (u^1,..,u^n)$ are the unknown functions of x and t representing the density and the velocity; $P = \kappa\rho^\gamma$, $\gamma \geq 1$, $\kappa > 0$, is the isentropic pressure; μ and λ are the viscosity coefficients, verifying

$$\mu > 0, \ \lambda - \mu/3 \geq 0. \tag{1.1}$$

The system (N.-S.) is solved subject to the initial conditions

$$(\rho(\cdot,0),u(\cdot,0))=(\rho_0(\cdot),u_0(\cdot)) \tag{1.2}$$

and one of the following boundary conditions.

Flows in $\mathbb{R}^n$: $\Omega=\mathbb{R}^n$, and for any $t\geq 0$, and $|x|\to+\infty$,

$$(\rho(x,t),u(x,t))\to(\bar{\rho},0), \tag{1.3}$$

for some $\bar{\rho}>0$. No-slip boundary conditions: for any $(x,t)\in\partial\Omega\times\mathbb{R}_+$,

$$u(x,t)=0. \tag{1.4}$$

Navier boundary conditions: for any $(x,t)\in\partial\Omega\times\mathbb{R}_+$ and $n=n(x)$ – the unit outer normal,

$$u\cdot n=0,\ ((\nabla u+\nabla u^t)n+Ku)_{\tan}=0,\ K\geq 0, \tag{1.5}$$

where $v_{\tan}=v-n(v\cdot n)$ is the component of vector v tangent to the boundary. The last condition is typically imposed on the common boundary between a fluid flow and a porous media. A non-negative coefficient K describes the porous material, see for example Beavers-Joseph [**1**]. The case $K=0$ corresponds to a perfectly lubricated boundary, Joseph [**18**]. We will assume in this notes that K is a constant.

1.2. Rankine-Hugoniot conditions. – In these notes we focus on discontinuous solutions of the Navier-Stokes equations. We start by considering the Rankine-Hugoniot conditions at the surface of the jump discontinuity of a weak solution (ρ,u). The following analysis can be found in Hoff [**10**] and Serre [**24**].

Let S be a smooth hypersurface in $\mathbb{R}^{n+1}$ and n be the normal to S, with space and time components n_x and n_t, and $|n_x|=1$. Let O be an open subset of $\mathbb{R}^{n+1}$, such that S divides O into two disjoint open sets $O^\pm$. Let (ρ,u) be a weak solution of Equations (N.-S.) in O with piece-wise smooth structure:

$$\rho\in C^{1,1}_{x,t}(\overline{O^\pm}),\quad u\in C^{2,1}_{x,t}(\overline{O^\pm}).$$

We denote by $[\![f]\!](z)=f^+(z)-f^-(z)$, $z=(x,t)\in S$, – the jump of function f across S at z. We assume that on S, $[\![\rho]\!]>0$ and $[\![u]\!]=0$. The later condition holds since we expect u to be continuous, due to the presence of viscosity in the momentum equation.

If we denote by $S_t=\{x\,:\,(x,t)\in S\}$ the section of S by a plane $t=\text{const.}$, then, from the continuity of u we conclude that $[\![\nabla u]\!]\tau=0$, for every τ in the tangent plane to S_t at $x\in S_t$. Since S_t is $n-1$ dimensional, $[\![\nabla u]\!]$ is a rank-1 matrix: there is $a\in\mathbb{R}^n$ such that

$$[\![\nabla u]\!]=an_x^t, \tag{1.6}$$

where n_x^t stands for the transpose of the column vector n_x. The Rankine-Hugoniot condition for the balance of mass equation is expressed as

$$n_t+u\cdot n_x=0, \tag{1.7}$$

meaning that velocity $(u(z),1)$ lies in the tangent plane to S at z. This implies that if $X^t(x,s)$ is a flow trajectory, i.e., the solution of the problem

$$\frac{dX^t}{dt} = u(X^t,t), \quad (X^s(x,s),s) \in S \cap O, \tag{1.8}$$

for some fixed $s \in \mathbb{R}$, then $(X^t(x,s),t) \in S$ for all times t while it remains in O. Thus, the interface of jump discontinuities S_t is transported by the flow.

The right-hand side of the momentum equation in (N.-S.) can be expressed as $\operatorname{div} \mathbb{S}$, where $\mathbb{S}$ is the Stokes stress tensor:

$$\mathbb{S} = \mu(\nabla u + \nabla u^t) + ((\lambda - \mu)\operatorname{div} u - P)\mathbb{I}.$$

Due to (1.7), the Rankine-Hugoniot conditions for the momentum equations are expressed as the continuity of the normal stress:

$$[\![\mathbb{S}]\!] n_x = 0.$$

Using the expression for the jump of ∇u from (1.6), we obtain the equation

$$\mu a + \mu(a \cdot n_x)n_x + [\![(\lambda - \mu)\operatorname{div} u - P]\!] n_x = 0,$$

which implies that $a = An_x$ for $A = -\frac{1}{2\mu}[\![(\lambda-\mu)\operatorname{div} u - P]\!]$ and from this we get $[\![\nabla u]\!] = An_x n_x^t$. Taking the trace in the last expression we find that

$$[\![(\lambda + \mu)\operatorname{div} u - P]\!] = 0, \tag{1.9}$$

and the formula for the jump of ∇u :

$$[\![\nabla u]\!] = \frac{[\![P]\!]}{\lambda + \mu} n_x n_x^t.$$

Note, that the last formula shows that $[\![\nabla u]\!]$ is symmetric, meaning that

$$[\![\operatorname{curl} u]\!] = 0.$$

Condition (1.9) is the first hint to a special role played by the function $F = (\lambda + \mu)\operatorname{div} u - P$, called the viscous flux. It has some extra regularity, compared to the regularity of ∇u and ρ.

We use (1.9) in the balance of mass equation $\partial_t \ln \rho + u \cdot \nabla \ln \rho + \operatorname{div} u = 0$ to find that

$$[\![\dot{\ln \rho}]\!] + \frac{1}{\lambda + \mu}[\![P]\!] = 0, \quad \dot{} = \partial_t + u \cdot \nabla,$$

due to the fact that S consists of the trajectories of the flow. For a typical pressure law $P = \kappa\rho^\gamma$, $\gamma \geq 1$, there are $C, c > 0$, depending on $\sup_{x,t} \rho$ and $\inf_{x,t} \rho > 0$, such that $c[\![\ln \rho]\!] \leq [\![P]\!] \leq C[\![\ln \rho]\!]$, and from the above equation we deduce bounds for the decay of $[\![\rho]\!]$:

$$[\![\ln \rho]\!](x,s)e^{-\frac{C}{\lambda+\mu}(t-s)} \leq [\![\ln \rho]\!](X^t(x,s),t) \leq [\![\ln \rho]\!](x,s)e^{-\frac{c}{\lambda+\mu}(t-s)}.$$

1.3. Flows with integrable inertia force. – The structure of surface S, discussed above, implies that the jump of the material acceleration $[\![\dot{u}]\!] = 0$. Since ∇u is discontinuous at S, with the jump given by (1.2), we expect $\dot{u}$ to be more regular than the derivatives in (x,t). This statement can be formalized by means of the energy estimates obtained by considering the convective derivative of the momentum equations, see Section 3.2. As we will see in that section, the energy estimates and the Sobolev imbedding theorem ensure that for any time $t > 0$ the inertia term is integrable with sufficiently high exponent:

$$\rho\dot{u}(\cdot,t) \in L^p(\Omega), \tag{1.10}$$

for some $p > n$. To obtain the information on ∇u, we consider the momentum equations as Lamé's equations for u:

$$\lambda\nabla \operatorname{div} u + \mu\Delta u = \nabla P + \rho\dot{u}, \tag{1.11}$$

and we also assume that $u(\cdot,t)$, $\nabla u(\cdot,t) \in L^2(\Omega)$, $P(\rho(\cdot,t)) \in L^\infty(\Omega)$.

1.3.1. *Interior regularity.* – Setting $F = (\lambda+\mu)\operatorname{div} u - P$, and taking divergence and curl of (1.11), we find that

$$\Delta F = \operatorname{div}(\rho\dot{u}), \quad \Delta \operatorname{curl} u = \operatorname{curl}(\rho\dot{u}).$$

Since $F(\cdot,t)$, $\operatorname{curl} u(\cdot,t) \in L^2(\Omega)$, $\rho\dot{u}(\cdot,t) \in L^p(\Omega)$, $p > n$, by the elliptic regularity and Sobolev's imbedding inequality, we get

$$\nabla F(\cdot,t),\ \nabla \operatorname{curl} u(\cdot,t) \in L^p_{\mathrm{loc}}(\Omega), \quad F(\cdot,t),\ \operatorname{curl} u(\cdot,t) \in C^\alpha_{\mathrm{loc}}(\Omega),\ \alpha = 1 - n/p.$$

This, in particular, shows that $\operatorname{div} u(\cdot,t)$ is in $L^\infty_{\mathrm{loc}}(\Omega)$. To get the information on other derivatives of u, we write Equations (1.11) as Poisson's equations

$$\mu\Delta u = \frac{\mu}{\lambda+\mu}\nabla P - \frac{\lambda}{\lambda+\mu}\nabla F + \rho\dot{u},$$

and using $\Gamma(x,y)$ – the fundamental solution of the Laplace equation on $\mathbb{R}^n$, we can write

$$u(x,t) = \frac{1}{\lambda+\mu}\int_{B_r(x_0)} \nabla_x\Gamma(x,y)P(y,t)\,dy + R(x,t), \tag{1.12}$$

for an arbitrary ball $B_r(x_0) \subset \Omega$ and the remainder term $R(\cdot,t) \in C^{1+\alpha}_{\mathrm{loc}}(\Omega)$.

From (1.12) we can find that $u(x,t)$ is log-Lipschitz continuous in x:

$$\sup_{x\neq y} \frac{|u(x,t) - u(y,t)|}{|x-y|\log^+|x-y|} < +\infty,$$

with $\log^+ r = \log(1 + r^{-1})$.

This estimate is sharp. In dimension 2, if $\rho(x,t)$ is piecewise C^α in x with jump discontinuity across a piecewise $C^{1+\alpha}$ curve that has a corner singularity at x_0 of the angle $\theta_0 \neq \{0, \pi/2, \pi\}$, for points x near x_0, $|\nabla u(x,t)|$ is of the order $\log^+|x - x_0|$.