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FOURIER ANALYSIS METHODS FOR COMPRESSIBLE FLOWS

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Abstract. – Since the 80's, Fourier analysis methods have known a growing importance in the study of linear and nonlinear PDE's. In particular, after the seminal paper by J.-M. Bony [4], the use of Littlewood-Paley decomposition and paradifferential calculus brought a number of new results whenever those equations are considered in the whole space or the torus.

We here aim at giving a survey of recent advances obtained by Fourier analysis methods in the context of compressible fluid mechanics. For simplicity, we shall focus on the *barotropic* compressible Navier-Stokes equations.

Introduction

This survey aims at presenting recent results in the theory of multi-dimensional compressible flows that have been obtained by means of elementary Fourier analysis. That approach is relevant in any context where a good notion of Fourier transform is available, and proved to be particularly efficient and robust for investigating a number of evolutionary fluid mechanics models. For simplicity, we shall concentrate on the following compressible Navier-Stokes equations governing the evolution of the density $\rho = \rho(t,x) \in \mathbb{R}^+$ and of the velocity field $u = u(t,x) \in \mathbb{R}^d$ of a barotropic viscous compressible fluid:

$$(1) \qquad \begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \mathrm{div}\,(\mu(\rho)(Du + \nabla u)) - \nabla(\lambda(\rho)\mathrm{div}\, u) + \nabla p = 0. \end{cases}$$

Above x belongs to the whole space $\mathbb{R}^d$ and the time variable t is nonnegative. The notation Du designates the *Jacobian matrix* of u (that is $(Du)_{ij} := \partial_j u^i$) while ∇u stands for the transposed matrix of Du (therefore $Du + \nabla u$ is twice the deformation tensor). In order to close the system, we assume that the pressure p to be a given

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(smooth) function P of the density (this is the *barotropic assumption*). The viscosity coefficients λ and μ are smooth functions of the density and satisfy the conditions

$$\alpha := \min\Big(\inf_{\rho>0}(\lambda(\rho)+2\mu(\rho)),\,\inf_{\rho>0}\mu(\rho)\Big)>0, \tag{2}$$

which ensures the second order operator in the momentum equation of (1) to be uniformly elliptic.

We supplement the system with initial data ρ_0 and u_0 at time $t=0$, and the conditions at infinity that u tends to 0 and that ρ tends to some positive constant (we shall take 1 for simplicity). The exact meaning of those boundary conditions will be given by the functional framework in which we shall solve the system.

In these notes we shall always consider the above system in the whole space $\mathbb{R}^d$ with $d \geq 2$, although our approach is adaptable to periodic boundary conditions $x \in \mathbb{T}^d$ and more generally $x \in \mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$ and so on. Let us also emphasize that our tools and methods are appropriate for investigating models with more physics (e.g., nonisothermal case, fluids endowed with internal capillarity and so on). For expository purpose, we here focus on (1).

These notes unfold as follows.

- The first section is devoted to presenting briefly the main techniques and tools that will be used to investigate (1). We first introduce the *Littlewood-Paley decomposition*, then define *Besov spaces* that are natural generalizations of the classical Sobolev spaces, and finally prove nonlinear estimates. In passing we present basic paradifferential calculus and give examples of estimates for linear equations that may be proved by taking advantage of Littlewood-Paley decomposition and paradifferential calculus. We chose to focus on the heat and Lamé equations, the transport equation and linear dispersive equations, which are frequently encountered when linearizing systems coming from fluid mechanics or physics.
- Solving (1) locally in time in *critical spaces* is the main purpose of the second section. As a warm up, we combine some results of the first section with the Banach fixed point theorem (or contracting mapping argument) so as to prove that the *incompressible* Navier-Stokes equations with small data, are globally well-posed. In passing, we introduce the notion of a *critical functional framework* for Partial Differential Equations. Next, we adapt this method to System (1) and get a local-in-time existence result in a critical functional framework. At first sight, it seems that uniqueness requires stronger assumptions than existence, a consequence of the fact that the proof *does not* rely on the contracting mapping argument but rather on high norm uniform bounds / low norm stability estimates like for solving quasilinear symmetric hyperbolic systems. In the last part of this section, we rewrite the system *in Lagrangian coordinates* so as to establish a second local-in-time existence result by means of the Banach fixed point theorem. With this second approach, we get existence, uniqueness

and continuity of the flow map (in Lagrangian coordinates) altogether, under the same regularity hypotheses as before.

- The third section is devoted to a fine analysis of the linearized system (1) about a stable constant state. We first obtain optimal regularity or decay estimates by taking advantage of the explicit formula for the solution. The second part of the section is somewhat disconnected of the study of compressible fluids. Here we present a general method, based on energy estimates, leading to optimal decay or regularity estimates for mixed-type linear systems. Typically, the systems that we shall consider are linear first order symmetric, with additional partially parabolic or dissipative terms (in the spirit of the work by Shizuta and Kawashima [**54**]). We shall show how an idea that is borrowed from Kalman controllability criterion may help to prove such estimates *without computing the solution explicitly.*
- The fourth section is devoted to the proof of global-in-time results for (1) supplemented with small initial data. Here the estimates of the third section play a key role. We first establish global well-posedness results in critical spaces and finally investigate the so-called incompressible limit. This latter result requires a finer analysis of the linearized system, pointing out its dispersive properties which are specific to the whole space case (in contrast with the other results).

We tried to keep those notes at an elementary level so as to give a general and as less technical as possible overview of how Fourier analysis techniques may be implemented. The reader may find more sophisticated results in e.g., [**2**], [**20**], [**22**] and in the references therein. For the sake of readability, most of the references and historical remarks are given at the end of Sections 2, 3 and 4.

1. The Fourier analysis toolbox

Here we introduce the Littlewood-Paley decomposition, define Besov spaces, establish product and composition estimates, and finally prove estimates for different types of linear PDEs that are frequently encountered when dealing with fluid mechanics models. More detailed proofs may be found in e.g., [**2, 20, 52, 56**].

1.1. A primer on Littlewood-Paley theory. – The Littlewood-Paley decomposition is a dyadic localization procedure in the frequency space for tempered distributions over $\mathbb{R}^d$. One of the main motivations for introducing such a localization when dealing with PDEs is that the derivatives act almost as homotheties on distributions with Fourier transform supported in a ball or an annulus.

In the L^2 framework, this noticeable property easily follows from Parseval's formula. The *Bernstein inequalities* below state that it remains true in any Lebesgue space:

Proposition 1.1 **(Bernstein inequalities).** – *For all* $0 < r < R$*, we have:*

- Direct Bernstein inequality*: a constant C exists so that, for any $k\in\mathbb{N}$, any couple (p,q) in $[1,\infty]^2$ with $q\geq p\geq 1$ and any function u of L^p with $\widehat{u}$ supported in the ball $B(0,\lambda R)$ of $\mathbb{R}^d$ for some $\lambda>0$, we have*
$$\|D^k u\|_{L^q}\leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}.$$
- Reverse Bernstein inequality*: there exists a constant C so that for any $k\in\mathbb{N}$, $p\in[1,\infty]$ and any function u of L^p with* $\operatorname{Supp}\widehat{u}\subset\{\xi\in\mathbb{R}^d\,/\,r\lambda\leq|\xi|\leq R\lambda\}$ *for some $\lambda>0$, we have*
$$\lambda^k\|u\|_{L^p}\leq C^{k+1}\|D^k u\|_{L^p}.$$

Proof. – Changing variables reduces the proof to the case $\lambda=1$. For proving the first inequality, we fix some smooth ϕ with compact support and value 1 over $B(0,R)$. One may thus write
$$\widehat{u}=\phi\,\widehat{u}\quad\text{whence}\quad D^k u=(D^k\mathcal{F}^{-1}\phi)\star u.$$
Therefore using convolution inequalities, one may write
$$\|D^k u\|_{L^q}\leq\|D^k\mathcal{F}^{-1}\phi\|_{L^r}\|u\|_{L^p}$$
with $1+1/q=1/p+1/r$ (here we need $q\geq p$), and we are done.

For proving the second inequality, we now assume that ϕ is compactly supported away from the origin and has value 1 over the annulus $\mathcal{C}(0,r,R)$. We thus have
$$\widehat{u}=\left(-i\frac{\xi}{|\xi|^2}\phi(\xi)\right)\cdot\widehat{\nabla u}\,(\xi).$$
Therefore, denoting by g the inverse Fourier transform of the first term in the r.h.s.,
$$\|u\|_{L^p}\leq\|g\|_{L^1}\|\nabla u\|_{L^p}.$$
This gives the result for $k=1$. The general case follows by induction. □

As solutions to nonlinear PDE's need not be spectrally localized in annuli (even if we restrict to initial data with this property), it is suitable to have a device which allows for splitting any function into a sum of functions with this spectral localization. This is exactly what Littlewood-Paley decomposition does.

To construct it, fix some smooth radial non increasing function χ with $\operatorname{Supp}\chi\subset B(0,\frac{4}{3})$ and $\chi\equiv1$ on $B(0,\frac{3}{4})$, then set $\varphi(\xi)=\chi(\xi/2)-\chi(\xi)$ so that
$$\chi+\sum_{j\in\mathbb{N}}\varphi(2^{-j}\cdot)=1\ \text{ in }\ \mathbb{R}^d\qquad\text{and}\qquad\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\cdot)=1\ \text{ in }\ \mathbb{R}^d\setminus\{0\}.$$
The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined by
$$\dot{\Delta}_j u:=\varphi(2^{-j}D)u:=\mathcal{F}^{-1}(\varphi(2^{-j}D)\mathcal{F}u):=2^{jd}h(2^j\cdot)\star u\quad\text{with}\quad h:=\mathcal{F}^{-1}\varphi.$$
We also introduce the low frequency cut-off operator $\dot{S}_j$:
$$\dot{S}_j u:=\chi(2^{-j}D)u:=\mathcal{F}^{-1}(\chi(2^{-j}D)\mathcal{F}u):=2^{jd}\check{h}(2^j\cdot)\star u\quad\text{with}\quad\check{h}:=\mathcal{F}^{-1}\chi.$$