TOPICS ON COMPRESSIBLE NAVIER-STOKES EQUATIONS: NON-DEGENERATE VISCOSITIES

Didier Bresch

Dedicated to the memory of Alexander Kazhikov.

1. Introduction

Professor A.V. Kazhikhov was one of the very distinguished analysts of our time. He made fundamental contributions on the mathematical theory of fluid mechanics, in particular he achieved outstanding results on the compressible Navier-Stokes system that, had, and still have, a significant influence on this field.

This special issue of Panoramas et Synthèses (SMF), dedicated to the memory of the Professor A.V. Kazhikhov, consists of three mathematical contributions dealing with compressible flows PDEs. They correspond to mini-courses respectively given by R. Danchin, A. Novtony, M. Perepetlisa with a constant or non-degenerate viscosities assumption. A contribution with degenerate viscosities (depending on the density) was presented by B. Desjardins during the session États de la Recherche but is not included in the volume because it does not really belongs to the same framework.

The main objective of this issue is to help the reader understand various mathematical tools that have been developed recently to deal with the problem of well posedness for the compressible Navier-Stokes equations with constant or at least non-degenerate viscosities. The three contributions focus on different kind of solutions, namely:

- Global weak solutions (regularity given by the energy estimates),
- Intermediate but not regular (discontinuous density and velocity with (low) regularity),
- Scaling invariant strong solutions (critical regularity).

In order to prepare the reader to these three main contributions, let us present in the following section different tools and main ingredients used by the authors. We hope by this section to motivate and help the reader start reading the contributions by showing them nice properties contained in the compressible Navier-Stokes equations. We start with the contribution by A. Novotny because it corresponds to the weakest regularity. We then continue with the contribution by M. Perepetlisa which concerns intermediate regularity, namely discontinuous density. Then we end the issue with the contribution by R. Danchin which focuses on critical spaces regularity. For each contributions, we will present some properties which are included in some sense in the contributions and mathematically justified in a more general setting sometimes.

Interested readers are invited to read the extensive exposition which can be found in the monographs by P.-L. Lions, E. Feireisl, A. Novotny and I. Straskraba, S. Benzoni and D. Serre, H.Bahouri, J.-Y. Chemin and R. Danchin and papers by D. Hoff *et al.*

2. Some comments on the contributions

The contribution by A. Novotny. – It concerns the existence of global weak solutions for the compressible Navier-Stokes-Fourier system in the spirit of the result obtained in 1933 by J. Leray for the incompressible Navier-Stokes equations, the existence of suitable weak solutions using the concept of relative entropies, the weak strong uniqueness property in the class of weak solutions. The main ingredients are energy estimates, weak limit, compactness of the temperature, div-curl Lemma, parametrized Young measures, effective flux, commutators, oscillations defect measure and renormalized continuity equation, relative entropy type inequality, weak-strong uniqueness techniques.

Global existence of weak solutions à la Leray. – Concerning the global in time existence result for the heat conducting Navier-Stokes equations, viscosities and conductivity coefficients are assumed to depend on the temperature (not on the density) and to be non-degenerate. In the barotropic case, namely in the non-dependent temperature case, viscosities are assumed to be constant and isotropic (same viscosities in all the directions: no eddy viscosities): see the book by P.-L. Lions and the one by A. Novotny and I. Strakraba. Extending the result to constant eddy viscosities is an interesting open problem with applications in geophysics.

For the heat conducting Navier-Stokes equations, dependency with respect to the temperature is possible because of the parabolic behavior of the equation satisfied by the temperature. Isotropy of the viscosities is necessary to get the weak compacness property on the effective flux $F = (2\mu + \lambda)\mathrm{div}u - p$. The regularity on the temperature helps to control commutators.

The form of the pressure is also strongly used to derive estimates on the density and find weak limit properties: the pressure is composed of two parts (a cold part which does not depend on the temperature but depends on the density and a part which depends on the density and on the temperature). It remains then to characterize weak limit of some difficult nonlinear terms: mainly the weak limit of the pressure term, viscous term. For that the authors show strong convergence of temperature. This point involves the treatment of the entropy production rate as a Radon measure and a convenient use of the compensated compactness lemma in combination with the theory of parametrized Young measures. Then it is needed to get strong convergence of densities. As for the barotropic case, weak compactness on the effective viscous flux is the key point and established using Riesz transform, commutators estimates à la Meyer through div-curl Lemma, boundedness of the oscillation defect measure for the sequence of densities, renormalized solution to the continuity equation and show the oscillations in the density sequence do not increase in time.

Some heuristic calculations. – For the reader's convenience, let us explain heuristically the main steps to prove global existence of weak solutions in the barotropic case with constant viscosities and power γ pressure law. This will help in order to understand the Navier-Stokes-Fourier analysis in the contribution by A. Novtony. In that case, the compressible Navier-Stokes equation with constant viscosities reads

$$(1)\qquad \begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla p(\rho) = \rho f, \end{cases}$$

with $p(\rho) = a\rho^\gamma$. For simplicity, we assume to be in a Lipschitz bounded domain Ω with homogeneous Dirichlet boundary conditions on the velocity. When ρ and u are regular and satisfied the continuity equation, for all $b \in \mathscr{C}([0,+\infty))$, it is clear, multiplying the mas equation by $b'(\rho)$, that (ρ, u) satisfy also the renormalized continuity equation (this terminology coming from the transport theory by R.J. DiPerna and P.-L. Lions). This equation reads

$$\partial_t b(\rho) + \mathrm{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho))\mathrm{div} u = 0.$$

For $T \in (0,+\infty)$, f, ρ_0, m_0 satisfying some technical assumptions, we say that a couple (ρ, u) is a weak renormalized solution with bounded energy if it possesses the following properties:

- $\rho \in L^\infty(0,T; L^\gamma(\Omega)) \cap \mathscr{C}^0([0,T], L^\gamma_{\mathrm{weak}}(\Omega))$, $\rho \geq 0$ *a.e.* in $(0,T)\times\Omega$, $\rho|_{t=0} = \rho_0$ *a.e.* in Ω,
- $u \in L^2(0,T; H^1_0(\Omega))$; $\rho|u|^2 \in L^\infty(0,T;L^1(\Omega))$ and $\rho u \in \mathscr{C}^0([0,T]; L^{2\gamma/(\gamma+1)}_{\mathrm{weak}}(\Omega))$, $(\rho u)|_{t=0} = m_0$ *a.e.* Ω;
- the solution extended by zero is solution of the mass and momentum equations in $\mathscr{D}'((0,T)\times R^d)$;
- for almost all $\tau \in (0,T)$, (ρ, u) satisfies the energy inequality
$$E(\rho,u)(\tau) + \int_0^\tau \int_\Omega (\mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div} u|^2) \leq E_0 + \int_0^t \int_\Omega \rho f \cdot u$$
In this inequality, $E(\rho,u)(\tau) = \int_\Omega (\rho|u|^2/2 + a\rho^\gamma/(\gamma-1))(\tau)$ denotes the total energy at time τ and $E_0 = \int_\Omega |m_0|^2/2\rho_0 + a\rho_0^\gamma/(\gamma-1))$ denotes the initial total energy;
- $b(\rho)$ satisfies the renormalized equation for b with some increasing properties.

The theory developed by P.-L. Lions to prove the global existence of renormalized weak solution with bounded energy asks for some limitation on the adiabatic constant γ, namely $\gamma > 3d/(d+2)$. Note that E. Feireisl *et al.* have generalized this approach in order to cover the range $\gamma > 3/2$ in dimension 3 and more generally $\gamma > d/2$ where d is the space dimension. We will try here to help the readers to understand the different lines of the proof in the "simplest" case, namely the one studied by P.-L. Lions. As usual in PDEs, the method asks for the compactness of a bounded sequence of solutions satisfying uniformly the energy estimates. The construction of such an approximate sequence is not the subject of this heuristic part (see the book by A. Novotny and I. Straskraba for details).

We present the proof due to P.-L. Lions and indicate quickly at the end how it was improved by E. Feireisl *et al.*. Let us note that using some appropriate multiplicator, for $\gamma > d/2$, it is possible to prove the following key estimate on the density

$$\int_0^\infty dt \int_\Omega \rho^{\gamma+\theta} \leq C(R,T) \text{ for } \theta = \frac{2}{d}\gamma - 1.$$

This observation is due to P.-L. Lions. To prove such an estimate, we need the fact that (ρ_n, u_n) is a sequence solution of the renormalized equation in which $b(s) = s^\theta$. Let us observe also that the regularity asked for the domain is linked to the use of the Bogovski operator as proposed by E. Feireisl *et al.* Using the energy estimate and the extra integrability property proved on the density, it is is then possible to precise some convergence and to show that

$$\begin{aligned} \rho_n &\to \rho \text{ in } \mathscr{C}^0([0,T]; L^\gamma_{\text{weak}}(\Omega)) \\ \rho_n^\gamma &\to \overline{\rho^\gamma} \text{ in } L^{(\gamma+\theta)/\gamma}((0,T)\times\Omega) \\ \rho_n u_n &\to \rho u \text{ in } \mathscr{C}^0([0,T]; L^{2\gamma/(\gamma+1)}(\Omega)) \\ \rho u_n^i u_n^j &\to \rho u u^j \text{ in } \mathscr{D}'((0,T)\times\Omega) \text{ for } i,j = 1,2,3. \end{aligned}$$

Let us note the convergence in some nonlinear terms which comes from the strong convergence of ρ_n deduced from the uniform estimate on $\partial_t \rho_n$ given by the mass equation and of the strong convergence of $\sqrt{\rho_n u_n}$ deduced form the uniform estimate on $\partial_t(\rho_n u_n)$ given by the momentum equation (which provides strong convergence in negative sobolev spaces) and by the convergence of the term $\int_0^T \int_\Omega \rho_n |u_n|^2$ writting $\int_0^T \int_\Omega \rho_n |u_n|^2 = \int_0^T <\rho_n u_n, u_n>_{H^{-1}(\Omega)\times H^1_0(\Omega)}$. Consequently, the prolongation by zero in $(0,T) \times R^3/\Omega$ of the functions $\rho, u, \overline{\rho^\gamma}$ denoted again $\rho, u, \overline{\rho^\gamma}$ satisfy the equations

$$\left\{ \begin{aligned} &\partial_t \rho + \operatorname{div}(\rho u) = 0, \\ &\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda+\mu)\nabla \operatorname{div} u + a \nabla \overline{\rho^\gamma} = \rho f. \end{aligned} \right. \tag{2}$$

The difficulty consists in proving that (ρ, u) is a renormalized weak solution with bounded energy and is mainly linked to the proof that $\overline{\rho^\gamma} = \rho^\gamma$ *a.e.* in $((0,T) \times \Omega$. This asks for a kind of compactness on the density sequence which is not available from the estimates only. This is an observation by P.-L. Lions which will fill the gap: The sequence $\{a\rho_n^\gamma - \lambda + 2\mu \operatorname{div} u_n\}_{n \in N^*}$, usually called viscous effective flux possesses a kind of weak compactness. This property was previously identified in one space dimension by D. Hoff and D. Serre. More precisely, we have the following property for all function $b \in \mathscr{C}^1([0,+\infty))$ satisfying some increasing properties at infinity

$$\lim_{n\to+\infty}\int_0^T\int_\Omega (a\rho_n^\gamma - (2\mu+\lambda)\mathrm{div}u_n)b(\rho_n)\varphi dxdt = \int_0^T\int_\Omega (a\overline{\rho^\gamma} - (2\mu+\lambda)\mathrm{div}u)\overline{b(\rho)}\varphi dxdt$$

where the overline quantities design the weak limit of the corresponding quantities and $\varphi \in \mathscr{D}((0,T)\times\Omega)$. In higher dimension, the proof by P.-L. Lions is based on harmonic analysis due to R. Coifman and Y. Meyer and takes into account the observations by D. Serre made in the one-dimensional case. The proof by E. Feireisl is based on div-curl Lemma introduced by F. Murat and L Tartar. Let us explain how P.-L. Lions concluded about the strong convergence on the density sequences using this weak compactness property on the effective flux. To simplify the computations, we will assume $\gamma \geq 3d/(d+2)$ and in that case due to the extra integrability on the density, we get that $\rho_n \in L^2((0,T)\times\Omega)$. In that case, the transport theory by R.J. DiPerna and P.-L. Lions applies to the continuity equation in order to get the validity of the renormalized equations. Then choose $b(s) = s\ln s$ in the renormalized formulation for ρ_n and ρ and do the difference of the two equations. Then pass to the limit $n \to +\infty$ and use the identity of weak compactness on the effective flux to replace terms with divergence of velocity by terms with density. This allows to get an evolution equation on the amplitude of the oscillation for the sequence of density. It reads

$$\partial_t(\overline{\rho\ln\rho} - \rho\ln\rho) + \mathrm{div}((\overline{\rho\ln\rho} - \rho\ln\rho)u) = \frac{a}{2\mu+\lambda}(\overline{\rho^\gamma}\rho - \overline{\rho^{\gamma+1}}).$$

The formal integration of this equation on $(0,T)\times\Omega$, the monotonicity of the pressure $p(\rho) = a\rho^\gamma$ and the strict-convexity of the function $s \mapsto s\ln s$, $s \geq 0$ implies that $\overline{\rho\ln\rho} = \rho\ln\rho$ if initially it is the case. In other terms, the weak convergence commute with a strictly convex function and therefore it gives the strong convergence of the densities. This achieves the proof of the Theorem in the case $\gamma \geq 3d/(d+2)$. The proof of E. Feireisl works even if the density is not *a priori* square integrable. For that E. Feireisl observes that it is possible to control the amplitude of the possible oscillations on the density in a norm L^p with $p > 2$ allowing to use an effective flux property with some troncature. The existence result is then also valid for $\gamma > d/2$: See the book by A. Novotny and I. Straskraba. For the heat conducting Navier-Stokes equations, the tools are more complicated but the steps endowed in the barotropic case are included in: Estimates and weak limits (step 1, step 4, step 5, step 6), Strong convergence of densities (Section 3) of course with not straightforward adaptations.

Remark. – If we have oscillation-concentration on the initial density sequence, the result is more complex because $\overline{\rho_0\ln\rho_0} \neq \rho_0\ln\rho_0$. We refer the reader to the recent paper by D. Bresch and M. Hillairet (2013) where they justify the derivation of viscous and compressible multi-component flow equations if initially the initial density sequence is uniformly bounded with corresponding Young measures which are linear convex combination of m Dirac measures.

Relative entropy and suitable weak solutions. – Since the pioneering work of C. Dafermos, relative entropy methods have become a crucial and widely used tool in the