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REAL RATIONAL SURFACES

by

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Abstract. – We survey recent results on real rational surfaces focusing on the links between their topology and their birational geometry.

***Résumé* (Surfaces rationnelles réelles).** – On survole les résultats récents sur les surfaces rationnelles réelles en insistant plus particulièrement sur les liens entre leur topologie et leur géométrie birationnelle.

1. Introduction

During the last decade [(1)], there were many progresses in the understanding of the topology of real algebraic manifolds, above all in dimensions 2 and 3. Results on real algebraic threefolds were addressed in the survey **[46]** with a particular emphasis on Kollár's results and conjectures concerning real uniruled and real rationally connected threefolds, see **[33]**, **[27, 26]**, **[14, 15]**, **[47]**. In the present paper, we will focus on real rational surfaces and especially on their birational geometry. Thus the three next sections are devoted to real rational surfaces; they are presented in a most elementary way. We state Commessatti's and Nash-Tognoli's famous theorems (Theorem 8 and Theorem 25). Among other things, we give a sketch of proof of the following statements:

– Up to isomorphism, there is exactly one single real rational model of each nonorientable surface (Theorem 13);

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[(1)] With the exception of some classical references, only references over the past years from the preceding "RAAG conference in Rennes," which took place in 2001, are included.

– The group of birational diffeomorphisms of a real rational surface is infinitely transitive (Theorem 15);
– The group of birational diffeomorphisms of a real rational surface X is dense in the group of $\mathcal{C}^\infty$-diffeomorphisms $\mathrm{Diff}(X(\mathbf{R}))$ (Theorem 27).

We conclude the paper with Section 5 devoted to a new line of research: the theory of *regulous functions* and the geometry we are able to define with them.

Besides the progresses in the theory of real rational surfaces, the classification of other real algebraic surfaces has considerably advanced during the last decade (see [**32**] for a survey): topological types and deformation types of real Enriques surfaces [**17**], deformation types of geometrically [(2)] rational surfaces [**19**], deformation types of real ruled surfaces [**58**], topological types and deformation types of real bielliptic surfaces [**13**], topological types and deformation types of real elliptic surfaces [**1, 6, 18**].

The present survey is an expansion of the preprint written by Johannes Huisman [**25**] from which we have borrowed several parts.

Convention. – In this paper, a *real algebraic surface* (resp. *real algebraic curve*) is a projective complex algebraic *manifold* of complex dimension 2 (resp. 1) endowed with an anti-holomophic involution whose set of fixed points is called *the real locus* and denoted by $X(\mathbf{R})$. A *real map* is a complex map commuting with the involutions. A *topological surface* is a real 2-dimensional $\mathcal{C}^\infty$-manifold. By our convention, a real algebraic surface X is nonsingular; as a consequence, if nonempty, the real locus $X(\mathbf{R})$ gets a natural structure of a topological surface when endowed with the euclidean topology. Furthermore $X(\mathbf{R})$ is compact since X is projective.

Acknowledgments. – Thanks to Daniel Naie for sharing his picture of the real locus of a blow-up, see Figure 1, to Jérémy Blanc for old references and the referee for useful remarks.

2. Real rational surfaces

2.1. Examples of rational surfaces. – A real algebraic surface X is *rational* if it contains a Zariski-dense subset real isomorphic to the affine plane $\mathbf{A}^2$. This is equivalent, as we shall see below, to the fact that the function field of X is isomorphic to the field of rational functions $\mathbf{R}(x,y)$. In the sequel, a rational real algebraic surface will be called a *real rational surface* for short and by our general convention, always assumed to be projective and nonsingular.

(2) See p. 12 before Theorem 17.

Example 1. – 1. The real projective plane $\mathbf{P}^2_{x:y:z}$ is rational. Indeed, each of the coordinate charts $U_0 = \{x \neq 0\}$, $U_1 = \{y \neq 0\}$, $U_2 = \{z \neq 0\}$ is isomorphic to $\mathbf{A}^2$. The real locus $\mathbf{P}^2(\mathbf{R})$ endowed with the euclidean topology is the topological real projective plane.

2. The product surface $\mathbf{P}^1_{x:y} \times \mathbf{P}^1_{u:v}$ is rational. Indeed, the product open subset $\{x \neq 0\} \times \{u \neq 0\}$ is isomorphic to $\mathbf{A}^2$. The set of real points $(\mathbf{P}^1 \times \mathbf{P}^1)(\mathbf{R}) = \mathbf{P}^1(\mathbf{R}) \times \mathbf{P}^1(\mathbf{R})$ is diffeomorphic to the 2-dimensional torus $\mathbf{S}^1 \times \mathbf{S}^1$ where $\mathbf{S}^1$ denotes the unit circle in $\mathbf{R}^2$.
3. The quadric $Q_{3,1}$ in the projective space $\mathbf{P}^3_{w:x:y:z}$ given by the affine equation $x^2+y^2+z^2 = 1$ is rational. Indeed, for a real point P of $Q_{3,1}$, denote by $T_P Q_{3,1}$ the real projective plane in $\mathbf{P}^3$ tangent to $Q_{3,1}$ at P. Then the stereographic projection $Q_{3,1} \setminus T_P Q_{3,1} \to \mathbf{A}^2$ is an isomorphism of real algebraic surfaces. For example in the case P is the North pole $N = [1:0:0:1]$, let $\pi_N\colon Q_{3,1} \to \mathbf{P}^2_{U:V:W}$ be the rational map given by

$$\pi_N\colon [w:x:y:z] \dashrightarrow [x:y:w-z]\ .$$

   Then π_N restricts to the stereographic projection from $Q_{3,1} \setminus T_N Q_{3,1}$ onto its image $\pi_N(Q_{3,1} \setminus T_N Q_{3,1}) = \{w \neq 0\} \simeq \mathbf{A}^2$.

   (The inverse rational map $\pi_N^{-1}\colon \mathbf{P}^2 \dashrightarrow Q_{3,1}$ is given by

$$\pi_N^{-1}\colon [x:y:z] \dashrightarrow [x^2+y^2+z^2 : 2xz : 2yz : x^2+y^2-z^2])\ .$$

   The real locus $Q_{3,1}(\mathbf{R})$ is the unit sphere $\mathbf{S}^2$ in $\mathbf{R}^3$.

To produce more examples, we recall the construction of the blow-up which is especially simple in the context of rational surfaces.

The blow-up $B_{(0,0)}\mathbf{A}^2$ of $\mathbf{A}^2$ at $(0,0)$ is the quadric hypersurface defined in $\mathbf{A}^2 \times \mathbf{P}^1$ by

$$B_{(0,0)}\mathbf{A}^2 = \{((x,y),[u:v]) \in \mathbf{A}^2_{x,y} \times \mathbf{P}^1_{u:v}\colon uy = vx\}.$$

The blow-up $B_{[0:0:1]}\mathbf{P}^2$ of $\mathbf{P}^2$ at $P = [0:0:1]$ is the algebraic surface

$$B_{[0:0:1]}\mathbf{P}^2 = \{([x:y:z],[u:v]) \in \mathbf{P}^2_{x:y:z} \times \mathbf{P}^1_{u:v}\colon uy - vx = 0\}.$$

The open subset $V_0 = \{((x,y),[u:v]) \in B_{(0,0)}\mathbf{A}^2\colon u \neq 0\}$ is Zariski-dense in $B_{(0,0)}\mathbf{A}^2$ and the map $\varphi\colon V_0 \to \mathbf{A}^2$, $((x,y),[u:v]) \mapsto (x, \frac{v}{u})$ is an isomorphism. Similarly, the open subset

$$\widetilde{U_2} = \{([x:y:z],[u:v]) \in B_{[0:0:1]}\mathbf{P}^2\colon z \neq 0, u \neq 0\}$$

is Zariski-dense in $B_{[0:0:1]}\mathbf{P}^2$ and the map $\widetilde{U_2} \to U_2 \simeq \mathbf{A}^2$,

$$([x:y:z],[u:v]) \mapsto [ux : v : uz]$$

is an isomorphism. Thus $B_{[0:0:1]}\mathbf{P}^2$ is rational. Now remark that the map $\varphi\colon V_1 = \{v \neq 0\} \to \mathbf{A}^2$, $((x,y),[u:v]) \mapsto (x, \frac{u}{v})$ is also an isomorphism and the surface $B_{(0,0)}\mathbf{A}^2$ is thus covered by two open subsets, both isomorphic to $\mathbf{A}^2$. We deduce that the surface $B_{[0:0:1]}\mathbf{P}^2$ is covered by the three open subsets $U_0, U_1, \widetilde{U_2} = B_{[0:0:1]}U_2 \simeq B_{(0,0)}\mathbf{A}^2$ hence covered by four open subsets, both isomorphic to $\mathbf{A}^2$. Up to affine transformation, we can define $B_P\mathbf{P}^2$ for any $P \in \mathbf{P}^2$ and it is now clear that the surface $B_P\mathbf{P}^2$ is covered by a finite number of open subsets, each isomorphic to $\mathbf{A}^2$. The same is clearly true for $\mathbf{P}^1 \times \mathbf{P}^1$. It is also true for $Q_{3,1}$. Indeed, choose 3 distinct real points P_1, P_2, P_3 of $Q_{3,1}$, and denote the open set $Q_{3,1} \setminus T_{P_i}Q_{3,1}$ by U_i, for $i = 1,2,3$. Since the common intersection of the three projective tangent planes is a single point, that, moreover does not belong to $Q_{3,1}$, the subsets U_1, U_2, U_3 constitute an open affine covering of $Q_{3,1}$.

Let X be an algebraic surface and P be a real point of X. Assume that P admits a neighborhood U isomorphic to $\mathbf{A}^2$ which is dense in X (by Corollary 12 below we have in fact that if X is rational, any real point of X has this property), and define the blow-up of X at P to be the real algebraic surface obtained from $X \setminus \{P\}$ and B_PU by gluing them along their common open subset $U \setminus \{P\}$. Then $B_PU \simeq B_PU_0$ is dense in B_PX and contains a dense open subset isomorphic to $U_0 \simeq \mathbf{A}^2$. At this point, we admit that this construction does neither depend on the choice of U, nor on the choice of the isomorphism between U and $\mathbf{A}^2$. See e.g., [**55**, §II.4.1] or [**46**, Appendice A] for a detailed exposition.

We get:

Proposition 2. – *Let X_0 be one of the surfaces $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ or $Q_{3,1}$. If*

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} X_0$$

is a sequence of blow-ups at real points, then X_n is a real rational surface.

Proof. – Indeed, from Example 1 and the comments above, any point $P \in X_i$ admits a neighborhood U isomorphic to $\mathbf{A}^2$ which is dense in X_i. □

Let $\pi\colon B_PX \to X$ be the blow-up of X at P. The curve $E_P = \pi^{-1}\{P\}$ is the *exceptional curve* of the blow-up. We say that B_PX is the blow-up of X at P and that X is obtained from B_PX by the *contraction* of the curve E_P.

Example 3. – Notice that if P is a real point of X, the resulting blown-up surface gets an anti-holomorphic involution lifting the one of X. If P is not real, we can obtain a real surface anyway by blowing up both P and $\overline{P}$: let U be an open neighborhood of P which is complex isomorphic to $\mathbf{A}^2(\mathbf{C})$ and define $B_{P,\overline{P}}X$ to be the result of the gluing of $X \setminus \{P, \overline{P}\}$ with both B_PU and $B_{\overline{P}}\overline{U}$.