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POSITIVE POLYNOMIALS AND SUMS OF SQUARES: THEORY AND PRACTICE

by

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In theory, theory and practice are the same.
In practice, they are different. — A. Einstein

Abstract. – If a real polynomial f can be written as a sum of squares of real polynomials, then clearly f is nonnegative on $\mathbb{R}^n$, and an explicit expression of f as a sum of squares is a certificate of positivity for f. This idea, and generalizations of it, underlie a large body of theoretical and computational results concerning positive polynomials and sums of squares. In this survey article, we review the history of the subject and give an overview of recent results, both theoretical results concerning the existence of certificates of positivity and work on computational and algorithmic issues.

Résumé **(Polynômes positifs et sommes de carrés : théorie et pratique).** – Un polynôme réel f qui est une somme de carrés de polynômes réels est clairement positif sur $\mathbb{R}^n$. L'écriture de f comme somme de carrés est donc un certificat de positivité pour f. En généralisant ce type de remarques, de nombreux résultats théoriques mais aussi calculatoires ont été obtenus concernant les polynômes positifs. Dans ce texte, on revient sur l'historique du sujet et on décrit les résultats récents, théoriques et algorithmiques, liés aux certificats de positivité.

If a real polynomial f in n variables can be written as a sum of squares of real polynomials, then clearly f must take only nonnegative values in $\mathbb{R}^n$. This simple, but powerful, fact and generalizations of it underlie a large body of theoretical and computational results concerning positive polynomials and sums of squares.

An explicit expression of f as a sum of squares is a *certificate of positivity for* f, i.e., a polynomial identity which gives an immediate proof of the positivity of of f on $\mathbb{R}^n$. In recent years, much work has been devoted to the study of certificates of positivity for polynomials. In this paper we will give an overview of some recent results in the theory and practice of positivity and sums of squares, with detailed references to the literature. By "theory," we mean theoretical results concerning the

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existence of certificates of positivity. By "practice," we mean work on computational and algorithmic issues, such as finding certificates of positivity for a given polynomial.

For the most part, we restrict results to those in a real polynomial ring. This is somewhat misleading, since it is impossible to prove most of the results for polynomials without using a more abstract approach. For example, in order to obtain a solution to Hilbert's 17th problem, it was necessary for Artin (along with Schreier) to first develop the theory of ordered fields! The reader should keep in mind that underneath the theorems in this paper lie the elegant and beautiful subjects of Real Algebra and Real Algebraic Geometry, among others.

The subject of positivity and sums of squares has been well-served by its expositors. There are a number of books and survey articles devoted to various aspects of the subject. Here we mention a few of these that the interested reader could consult for more details and background on the topics covered in this paper, as well as related topics that are not included: There are the books by Prestel and Delzell [**69**] and Marshall [**44**] on positive polynomials, a survey article by Reznick [**74**] about psd and sos polynomials with a wealth of historical information, and a recent survey article by Scheiderer [**83**] on positivity and sums of squares which discusses results up to about 2007. Finally, there is a survey article by Laurent [**42**] which discusses positivity and sums of squares in the context of applications to polynomial optimization.

1. Preliminaries and background

In this section, we introduce the basic concepts and review some of the fundamental results in the subject, starting with results in the late 19th century. For a fuller account of the historical background, see the survey [**74**]. For a more detailed survey of the subject up to about 2007, readers should consult the survey article [**83**].

1.1. Notation. – Throughout, we fix $n \in \mathbb{N}$ and let $\mathbb{R}[X]$ denote the real polynomial ring $\mathbb{R}[X_1, \ldots, X_n]$. We denote by $\mathbb{R}[X]^+$ the set of polynomials in $\mathbb{R}[X]$ with nonnegative coefficients. The following monomial notation is convenient: For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let X^α denote $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. For a commutative ring A, we denote the set of sums of squares of elements of A by $\sum A^2$.

We define the basic objects studied in real algebraic geometry. Given a set G of polynomials in $\mathbb{R}[X]$, the closed semialgebraic set defined by G is

$$\mathcal{S}(G) := \{x \in \mathbb{R}^n \mid g(x) \geq 0 \text{ for all } g \in G\}.$$

If G is finite, $\mathcal{S}(G)$ is the basic closed semialgebraic set generated by G.

The basic algebraic objects of interest are defined as follows. For a finite subset $G = \{g_1, \ldots, g_r\}$ of $\mathbb{R}[X]$, the preordering generated by G is

$$PO(G) := \{ \sum_{e=(e_1,\ldots,e_r)\in\{0,1\}^r} s_e g_1^{e_1} \ldots g_r^{e_r} \mid \text{ each } s_e \in \sum \mathbb{R}[X]^2 \}.$$

The quadratic module generated by G is

$$M(G) := \{s_0 + s_1 g_1 + \cdots + s_r g_r \mid \text{ each } s_i \in \sum \mathbb{R}[X]^2\}.$$

Notice that if $f \in PO(G)$ or $f \in M(G)$, then f is clearly positive on $\mathcal{S}(G)$ and an identity $f = \sum_{e \in \{0,1\}^r} s_e g_1^{e_1} \dots g_r^{e_r}$ or $f = s_0 + s_1 g_1 + \cdots + s_r g_r$ is a certificate of positivity for f on $\mathcal{S}(G)$.

Traditionally, a result implying the existence of certificates of positivity for polynomials on semialgebraic sets is called a *Positivstellensatz* or a *Nichtnegativstellensatz*, depending on whether the polynomial is required to be strictly positive or non-strictly positive on the set. We will use the term "representation theorem" for any theorem of this type and refer to a "representation of f" (as a sum of squares, in the preordering, etc.), meaning an explicit identity for f.

1.2. Classic results. – A polynomial $f \in \mathbb{R}[X]$ is positive semidefinite, psd for short, if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. We say f is sos if $f \in \sum \mathbb{R}[X]^2$. Of course, f sos implies that f is psd, and for $n = 1$, the converse follows from the Fundamental Theorem of Algebra.

We begin our story in 1888, when the 26-year-old Hilbert published his seminal paper on sums of squares **[25]** in which he showed that for $n \geq 3$, there exist psd forms (homogenous polynomials) in n variables which are not sums of squares. In the same paper, he proved that every psd ternary quartic – homogenous polynomial of degree 4 in 3 variables – is a sum of squares. (1) Hilbert was able to prove that for $n = 3$, every psd form is a sum of squares of rational functions, but he was not able to prove this for $n > 2$. This became the seventeenth on his famous list of twenty-three mathematical problems that he announced at the 1900 International Congress of Mathematicians in Berlin. In 1927, E. Artin **[2]** settled the question:

***Theorem 1* (Artin's Theorem).** – *Suppose $f \in \mathbb{R}[X]$ is psd, then there exists nonzero $g \in \mathbb{R}[X]$ such that $g^2 f$ is sos.*

The following Positivstellensatz has until recently been attributed to Stengle **[93]**, who proved it in 1974. It is now known that the main ideas were in a paper of Krivine's from the 1960's.

***Theorem 2* (Classical Positivstellensatz).** – *Suppose $S = \mathcal{S}(G)$ for finite $G \subseteq \mathbb{R}[X]$ and $f \in \mathbb{R}[X]$ with $f > 0$ on S. Then there exist $p, q \in PO(G)$ such that $pf = 1 + q$.*

(1) Hilbert worked with forms, however for the purposes of this paper we prefer to work in a non-homogenous setting. A form can be dehomogenized into a polynomial in one less variable and the properties of being psd and sos are inherited under dehomogenization. When discussing work related to Hilbert's work, we will use the language of forms, otherwise, we state results in terms of polynomials.

1.3. Bernstein's and Pólya's theorems. – Certificates of positivity for a univariate $p \in \mathbb{R}[x]$ such that $p \geq 0$ or $p > 0$ on an interval $[a, b]$ have been studied since the late 19th century. Questions about polynomials positive on an interval come in part from the relationship with the classic Moment Problem, in particular, Hausdorf's solution to the Moment Problem on $[0, 1]$ [**24**].

In 1915, Bernstein [**6**] proved that if $p \in \mathbb{R}[x]$ and $p > 0$ on $(-1, 1)$, then p can be written as a positive linear combination of polynomials $(1 - x)^i(1 + x)^j$ for suitable integers i and j; however, it might be necessary for $i + j$ to exceed the degree of p. Notice that writing p as such a positive linear combination is a certificate of positivity for p on $[-1, 1]$.

Pólya's Theorem, which he proved in 1928 [**58**], concerned forms positive on the standard $n-1$-simplex $\Delta_{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_i x_i = 1\}$.

***Theorem 3* (Pólya's Theorem).** – *Suppose $f \in \mathbb{R}[X]$ is homogeneous and is strictly positive on Δ_{n-1}, then for sufficiently large N, all of the coefficients of $(X_1 + \cdots + X_n)^N f$ are positive.*

Here "all coefficients are positive" means that every monomial of degree $\deg f + N$ appears with a strictly positive coefficient.

Bernstein's result is equivalent to the one-variable dehomogenized version of Pólya's Theorem: If $p \in \mathbb{R}[x]$ is positive on $(0, \infty)$, then there exists $N \in \mathbb{N}$ such that $(1+x)^N p$ has only positive coefficients. The equivalence is immediate by applying the "Goursat transform" which sends p to

$$(x + 1)^d p\left(\frac{1 - x}{1 + x}\right),$$

where $d = \deg p$.

1.4. Schmüdgen's Theorem and beyond. – In 1991, Schmüdgen [**88**] proved his celebrated theorem on representations of polynomials strictly positive on compact basic closed semialgebraic sets. This result began a period of much activity in Real Algebraic Geometry, which continues today, and stimulated new directions of research.

***Theorem 4* (Schmüdgen's Positivstellensatz).** – *Suppose G is a finite subset of $\mathbb{R}[X]$ and $\mathcal{S}(G)$ is compact. If $f \in \mathbb{R}[X]$ is such that $f > 0$ on $\mathcal{S}(G)$, then $f \in PO(G)$.*

Schmüdgen's theorem yields "denominator-free" certificates of positivity, in contrast to Artin's theorem and the Classic Positivstellensatz. The underlying reason that such certificates exist is that the preordering $PO(G)$ in this case is *Archimedean*: Given any $h \in \mathbb{R}[X]$, there exists $N \in \mathbb{N}$ such that $N \pm h \in PO(G)$. Equivalently, there is some $N \in \mathbb{N}$ such that $N - \sum X_i^2 \in PO(G)$. It is a fact that if $\mathcal{S}(G)$ is compact, then $PO(G)$ is Archimedean. This follows from Schmüdgen's proof of his theorem; there is a direct proof due to Wörmann [**95**].

The definition of Archimedean for a quadratic module M is the same as for a preordering. If $M(G)$ if Archimedean, then it is immediate that $\mathcal{S}(G)$ is compact;