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A SURVEY ON O-MINIMAL STRUCTURES

by

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Abstract. – We analyze some aspects of the theory of o-minimal structures, and its applications to various contexts, from differential equations to diophantine geometry. In particular, we illustrate on various examples several analytic and geometric methods involved in the proofs of o-minimality.

Résumé **(Une synthèse des structures o-minimales).** – Nous considérons divers aspects de la théorie des structures o-minimales, et ses applications à différents domaines, depuis les équations différentielles jusqu'à la géométrie diophantienne. En particulier, nous illustrons plusieurs méthodes analytiques et géométriques développées dans les preuves d'o-minimalité.

What are o-minimal structures, what is o-minimal geometry? In his course on o-minimal geometry [**8**], M. Coste says that "*the main feature of o-minimal structures is that there are no "monsters" in such structures*". As an example of "monster," he mentions the closure of the graph Γ of the function $x \mapsto \sin(1/x)$ for $x > 0$, which is connected, but not arcwise connected. One also observes that, because of the oscillating nature of the set Γ, the intersection of Γ with the positive real axis has infinitely many connected components. Roughly speaking, o-minimal structures are intended to provide a setting for "tame geometry" in which such bad things cannot happen.

Two examples of such tame geometries are well-known: the *semialgebraic* and the *subanalytic* geometry. They both satisfy many finiteness properties. For example, every semialgebraic set has finitely many connected components, which are semialgebraic as well. A careful examination of the proof of these properties shows that they follow from a few simple axioms. These axioms have been identified by L. van den Dries in 1984 [**10**], and collected under the terminology of *theory of finite type.* During the same year, stimulated by van den Dries' work, A. Pillay and C. Steinhorn

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extended this study using the terminology *o-minimal theory* [**59**]. The main problems considered in this domain of research are the following:

1. What are the properties of the collections of sets which satisfy the axioms of o-minimality?
2. Beside the semialgebraic and the subanalytic frameworks, are there many families of sets which satisfy these axioms?
3. What kind of applications can we expect from this study?

Our goal in this survey is to summarize several answers that have been given to these questions.

Among the sources of o-minimal geometry, we can quote A. Tarski's monograph on the theory of the structure $(\mathbb{R}, +, \cdot)$ [**76**]. This inspiration is followed by S. Lojasiewicz' work on semianalytic sets [**44**], as well as A. Gabrielov's celebrated theorem of the complement of a subanalytic set [**24**], and subsequently H. Hironaka's rectilinearization of subanalytic sets [**28**]. It is also worth mentioning the title of the fifth section of A. Grothendieck's "Esquisse d'un programme": *Haro sur la topologie dite "générale", et réflexions heuristiques vers une topologie dite "modérée"*[1] [**26**], which gave rise to the terminology "tame geometry," or "tame topology".

This important topic, on the border between model theory and geometry, has been the subject of an impressive body of literature. Several excellent surveys and books are devoted to it. Let us mention van den Dries' book [**12**], M. Coste's monograph [**8**] and C. Miller's and van den Dries' article [**18**]. We should also cite M. Shiota's book [**70**], which contains a theory close to o-minimality, namely the theory of $\mathfrak{X}$-*sets* (see Remark 1.8), and contains also the proof of several nice results (see Section 3.7).

In Section 1 we recall the definition of an o-minimal structure from two points of view: the *geometric* one, and the *model-theoretic* one. In Section 2, we give a few explanations about the seminal works [**10, 59**]. The main classical properties of o-minimal structures are listed in Section 3. Several examples of o-minimal structures are described in Section 4. Section 5 is devoted to recent applications of o-minimality in diophantine geometry. We give some details in Section 6 about the techniques which have been employed in proving o-minimality. The last section 7 contains open problems.

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1. Basic definitions

There are traditionally two ways to introduce o-minimality: the geometric one and the model-theoretic one. We summarize both of them in this section.

[1] *Denunciation of so-called "general" topology, and heuristic reflections towards a so-called "tame" topology.*

1.1. The geometric point of view

Definition 1.1. – A *structure expanding the real field* $\mathbb{R}$ is a collection $\mathcal{S} = (S_n)_{n\in\mathbb{N}}$, where each S_n is a set of subsets of the affine space $\mathbb{R}^n$, satisfying the following properties:

1. All algebraic subsets of $\mathbb{R}^n$ are in S_n.
2. For every n, S_n is a Boolean subalgebra of the powerset of $\mathbb{R}^n$.
3. If $A \in S_m$ and $B \in S_n$, then $A \times B \in S_{m+n}$.
4. If $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates and $A \in S_{n+1}$, then $p(A) \in S_n$.

The elements of S_n ($n \in \mathbb{N}$) are called the *definable sets* of the structure. In particular, if $\mathcal{F}$ is a collection of functions $f\colon \mathbb{R}^m \to \mathbb{R}$, for various $m \in \mathbb{N}$ (resp. of subsets of $\mathbb{R}^m$ for various $m \in \mathbb{N}$), the *structure* $\mathbb{R}_{\mathcal{F}}$ *generated by* $\mathcal{F}$ (or the *expansion of* $\mathbb{R}$ *by the elements of* $\mathcal{F}$) is the smallest structure expanding the real field which contains the graphs of all functions in $\mathcal{F}$ (resp. all the elements of $\mathcal{F}$ if $\mathcal{F}$ is a collection of sets).

Finally, a map $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^p$ is called *definable* if its graph is definable (which implies that the domain A is a definable subset of $\mathbb{R}^n$).

The notion of o-minimality is related to a specific property of the definable subsets of $\mathbb{R}$:

Definition 1.2. – A structure $\mathcal{S}$ is called *o-minimal* if the elements of S_1 are all the finite unions of singletons and intervals.

Actually, instead of the real field, in the definitions above we could have considered any real closed field. But, since our goal is to insist on the applications of o-minimality for which this degree of generality is not relevant (such as real analytic dynamical systems, quasianalytic algebras of $\mathcal{C}^\infty$ real functions, diophantine geometry on complex algebraic varieties), this more general point of view will not be developed in the next sections.

The following notion plays an important role in o-minimality:

Definition 1.3. – A structure $\mathcal{S}$ is *model complete* if, in the above definition of definable set, the operation of taking complements is superfluous.

Example 1.4. – The smallest o-minimal expansion of the real field is the structure $\mathbb{R}_\varnothing$ (or $\mathbb{R}_{\text{alg}}$), whose elements are the semialgebraic subsets of the affine spaces $\mathbb{R}^n, n \in \mathbb{N}$. This is an immediate consequence of the celebrated "Tarski-Seidenberg theorem" (which states that if $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates and $A \in \mathbb{R}^{n+1}$ is a semialgebraic set, then $p(A)$ is semialgebraic), and the fact that the semialgebraic subsets of the real line are precisely the finite unions of points and intervals [(2)].

The following definition generalizes the classical notions of semialgebraic or semi-analytic sets:

Definition 1.5. – Given a collection $\mathscr{F}$ of functions as in Definition 1.1, a subset of $\mathbb{R}^n$ of the form:

$$\{x \in \mathbb{R}^n : P(x, f_1(x), \dots, f_k(x)) = 0\},$$

where $P \in \mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_k]$ and $f_1, \dots, f_k$ belong to $\mathscr{F}$, is called a $\mathscr{F}$-set.

The next simple proposition is a frequently used in the proof of o-minimality (see Section 6):

Proposition 1.6. – *Consider a collection $\mathscr{F}$ of functions $f : \mathbb{R}^m \to \mathbb{R}$, for various $m \in \mathbb{N}$ such that:*

(i) *the structure $\mathbb{R}_{\mathscr{F}}$ is model complete;*
(ii) *every $\mathscr{F}$-set has finitely many connected components.*

Then $\mathbb{R}_{\mathscr{F}}$ is o-minimal.

Proof. – We actually prove that, given any definable set $A \subseteq \mathbb{R}^m$, there exist $n \in \mathbb{N}$ and a $\mathscr{F}$-set $B \subseteq \mathbb{R}^{m+n}$ such that $A = \Pi_m(B)$ (where Π_m is the projection onto the first m coordinates), and hence has only finitely many connected components. We proceed by an induction on the "complexity" of the definable sets.

1. Every semialgebraic set is obviously a $\mathscr{F}$-set.

2. Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be two projections of $\mathscr{F}$-sets: there exist $p, q \in \mathbb{N}$, two polynomials P in $m+p$ variables and Q in $n+q$ variables, and two collections $f_1, \dots, f_p$ and $g_1, \dots, g_q$ of elements of $\mathscr{F}$ such that $A = \Pi_m(S)$ and $B = \Pi_n(T)$, with:

$$S = \left\{(x, y) \in \mathbb{R}^{m+p} : P(x, y, f_1(x, y), \dots, f_p(x, y)) = 0\right\} \text{ and}$$
$$T = \left\{(z, t) \in \mathbb{R}^{m+q} : Q(z, t, g_1(z, t), \dots, g_q(z, t)) = 0\right\}.$$

Suppose first that $m = n$. Then $A \cup B$ is the projection onto the first m coordinates of the $\mathscr{F}$-set

$$\left\{(x, y, t) \in \mathbb{R}^{m+p+q} : P(x, y, f_1(x, y), \dots, f_p(x, y)) \cdot Q(x, t, g_1(x, t), \dots, g_q(x, t)) = 0\right\},$$

and $A \cap B$ is the projection onto the first m coordinates of the $\mathscr{F}$-set

$$\{(x, z, t) \in \mathbb{R}^{m+p+q} : P(x, y, f_1(x, y), \dots, f_p(x, y))^2 + Q(x, t, g_1(x, t), \dots, g_q(x, t))^2 = 0\}.$$

(2) The names Tarski and Seidenberg are traditionally associated for this result. In fact, A. Tarski gave a first proof in [**76**] in 1951. Later, in 1954, A. Seidenberg gave a new proof in [**69**]. The technical difference between the two proofs is explained in Seidenberg's article, p. 374, additional remark (e): in Tarski's method, Sturm's Theorem is used to eliminate each variable, whereas in Seidenberg's proof, Sturm's theorem is used only once, to eliminate the last variable.