Analytic arcs and Real Analytic Singularities

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ANALYTIC ARCS AND REAL ANALYTIC SINGULARITIES

by

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Dedicated to Marie-Françoise, Michel and Louis.

Abstract. – The aim of this survey is to present recent development in real singularities with emphasis on new methods of classification of real analytic function germs. The concepts of a blow-analytic or of an arc-analytic equivalence have emerged back in the 80's of the last century, however an important progress has been made in last 15 years thanks to the new methods inspired by the motivic integration approach. Application of this idea to the real context has been made possible in particular due to the introduction of arc symmetric sets, virtual Betti numbers and the virtual Poincaré polynomial. These new techniques have been then used to construct various subtle invariants of real analytic function germs.

***Résumé* (Arcs analytiques et singularités analytiques réelles).** – On présente dans ce texte les avancées récentes en théorie des singularités réelles. On donne notamment de nouvelles méthodes de classification des germes de fonctions analytiques réelles. Les notions d'équivalences blow-analytique et arc-analytique sont apparues dans les années 80 du siècle dernier. L'utilisation de nouvelles méthodes inspirées de la théorie de l'intégration motivique, a donné, pendant ces 15 dernières années, des résultats majeurs portant sur ces 2 notions d'équivalence. Cette nouvelle approche utilise la théorie des ensembles symétriques par arcs ainsi que les notions de nombres de Betti virtuels et de polynôme de Poincaré virtuel. Ces techniques produisent de nouveaux invariants fins pour les germes de fonctions analytiques réelles.

1. Introduction

The goal of this survey is to present some aspects of the recent progress in the theory of real analytic singularities with particular emphasis on the use of analytic arcs and the structure of the spaces of such arcs.

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The famous Curve Selecting Lemma of Bruhat-Cartan-Wallace has appeared already in the 1950's and later was made by S. Łojasiewicz one of the most powerful tools in the real analytic geometry. The idea of studying the structure of the space of germs of analytic or formal arcs on singular complex varieties appeared in a preprint of J.F. Nash (1965), which was published 30 years later [**38**]. The motivation of J.F. Nash was to understand the resolution of singularities of H. Hironaka or possibly to give an alternative approach to this fundamental problem.

Analytic arcs also played an important role in the classification of real analytic function germs. A fundamental question in the classification of the germs of real analytic functions is finding a "good" definition of their equivalence. For two germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of analytic functions, we would like to say that they are *equivalent* if there exists the germ of a homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ h$. If we just ask h to be a homeomorphism the equivalence relation is too coarse and ignores the metric and analytic differences between f and g. On the other hand when we restrict the family h to C^1 (or even bi-Lipschitz) maps, the resulting equivalence is too fine in the sense that it admits continuous moduli: there are analytic families, in finitely many parameters, of analytic germs such that every two distinct germs are not C^1 (even bi-Lipschitz) equivalent. In the 70's T.C. Kuo introduced the notion of blow-analytic map. Given real analytic manifolds M, N, he called a mapping $f : M \to N$ *blow-analytic* if there exists $\sigma : \tilde{M} \to M$, a locally finite composition of blowing-ups with smooth centers, such that $f \circ \sigma$ is analytic. If we impose the condition that h and h^{-1} are blow-analytic we get *blow-analytic equivalence* (in the broadest sense). T.C. Kuo [**28**] proved that the blow-analytic equivalence classes of an analytic family of real analytic function germs with isolated singularities have no moduli. So the blow-analytic equivalence seems to be most natural for the classification of germs of real analytic functions. Kuo's idea generated a lot of interesting results towards better understanding the blow-analytic maps, construction of blow-analytic invariants, classification of certain families of analytic germs.

In general it is not obvious to check that a given map is blow-analytic. In the 80's a more general notion of arc-analytic function was introduced by the author [**29**]. Let M, N be real analytic manifolds, a map $f : M \to N$ is called *arc-analytic* if $f \circ \gamma$ is analytic for every analytic arc γ. In Section 2 I will give a detailed introduction to arc-analytic functions and arc-symmetric sets.

Clearly each blow-analytic map is arc-analytic (and subanalytic). In the semialgebraic setting the inverse was established by E. Bierstone and P.D. Milman [**3**], and also by A. Parusiński [**40**]. However in the general analytic case the question whether every subanalytic and arc-analytic function is blow-analytic remains a challenging open problem. I will discuss this issue in Section 5. Lipschitz properties of arc- and blow- analytic functions are discussed in this section.

In the late 90's M. Kontsevich brought new ideas of the motivic measure to study the structure of the space of analytic (or formal) arcs. This sparkling idea was then developed by J. Denef and F. Loeser [**11**]. This was an important breakthrough which has numerous applications in algebraic geometry and singularity theory. It turns out to be also surprisingly useful and applicable in the real context. I will describe in this survey some important results obtained using motivic integration in the real analytic case. In January 2003 a *Winter School Real Algebraic and Analytic Geometry & Motivic Integration* was held in Aussois, the lectures were published in [**10**]. The goal was to introduce motivic integration methods to the community of real algebraic and analytic geometers. Looking back from the eight-year perspective, I am convinced that it was a successful investment.

Indeed, to understand blow-analytic (or arc-analytic) equivalence it is natural to study the induced maps on the corresponding spaces of analytic arcs. However the spaces of arcs are of infinite dimension, so there very few tools available. When applying motivic integration methods one attaches to the germ of an analytic function a sequence of subsets in finite dimensional spaces (of polynomial arcs). To describe the fine properties of these sets McCrory and Parusiński [**36**], Fichou[**13**], introduced the virtual Betti numbers and virtual Poincaré polynomial for a large class of subsets of finite dimensional affine space, including algebraic and also arc-symmetric semi-algebraic sets. I will describe this topic in Section 3. In fact these notions are interesting in their own right in real algebraic geometry. The (virtual) invariants of those sets plays a crucial role in the construction of blow-analytic invariants of the germs of analytic functions. These applications are described in Section 4, where will describe a complete blow-analytic classification of the germs of real analytic functions in 2 variables, due to S. Koike and A. Parusiński [**26, 27**].

Arc-analytic phenomena appear in the theory of hyperbolic polynomials with analytic coefficients and analytic families of symmetric matrices. A classical result of F. Rellich [**47**] states that any one parameter real analytic family of symmetric matrices admits a simultaneous analytic diagonalization, so in particular the eigenvalues are analytic functions of the parameter. In Section 6 I will describe recent joint results with L. Paunescu [**33**] which generalizes Rellich's theorem to the case of several parameters. We have shown that after a suitable blowing-ups of the space of parameters the Rellich theorem holds locally. Moreover the ordered eigenvalues (in several parameters) are Lipschitz functions.

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2. Arc-symmetric subsets of affine space

This section contains essentially the background on analytic arcs, arc-symmetric sets and arc-analytic functions. Details can be found in [**29**],[**30**]. For basic facts on real algebraic and semi-algebraic sets we refer to [**5**].

2.1. Analytic arcs. – By *an analytic arc* in an analytic manifold M we mean an analytic non-constant mapping $\gamma : (-\varepsilon, \varepsilon) \to M$, for some $\varepsilon > 0$. The image of the germ of γ at 0 is well-defined. Indeed, γ as a mapping to a neighborhood of $\gamma(0)$, is finite and proper, (cf. [**35**]). The image of the germ of γ may be of two topological types.

If γ is injective, then the image of the germ of γ is an irreducible analytic germ of dimension 1, which is homeomorphic to the germ of the interval $(-\alpha, \alpha)$ at 0. If γ is not injective (in a neighborhood of 0), then $\gamma(t) = \eta(t^k)$, where $k = 2^d$ and η is an analytic injective arc. So in this case the image of the germ of γ is homeomorphic to the germ of $[0, \alpha)$ at 0, it is a half-branch of an irreducible analytic germ of dimension 1. What is crucial for our theory is the fact that one half-branch determines the other.

Later on we shall not distinguish between an arc and its germ at 0. In the sequel we will work rather with the images of analytic arcs (by abuse of language we will sometimes also call them analytic arcs). To explain when two injective analytic arcs have the same image we define an equivalence relation on the space of all (not necessarily injective) analytic arcs.

We say that two arcs γ and γ' are *equivalent* if there exist: a germ of analytic arc η, and h, h' two germs of analytic homeomorphisms of neighborhoods of $0 \in \mathbb{R}$ such that

$$\gamma = \eta \circ h \qquad \textit{and} \qquad \gamma' = \eta \circ h'. \tag{1}$$

For instance $\gamma(t) = (t^2, t^3)$ and $\gamma' = (t^6, t^9)$ are equivalent, where $\eta = \gamma$, $h(t) = t$, $h'(t) = t^3$. Note that, in general, an analytic homeomorphism need not have an analytic inverse. This is an equivalence relation, but the transitivity is not obvious.

One of the most powerful tool in the real algebraic (more generally analytic) geometry is the Curve Selection Lemma of Bruhat, Cartan and Wallace. In the real algebraic case it can be stated as follows (see eg.[**5**], [**9**]): let $A \subset \mathbb{R}^n$ be semialgebraic set, suppose that $a \in \mathbb{R}^n$ is an accumulation point of A, that is $a \in \overline{A \setminus \{a\}}$, then there exists a continuous arc $\lambda : [0,1) \to \mathbb{R}^n$ with semialgebraic graph, such that $\lambda(0) = a$ and $\lambda(0,1) \subset A$. In fact, by the classical Puiseux theorem (see eg. [**35**], [**51**]), we can say more: there exists an integer k such that the function $t \mapsto \lambda(t^k) = \gamma(t)$ admits an analytic extension in a neighborhood of $0 \in \mathbb{R}$. More precisely we can write $\gamma(t) = \tilde{\gamma}(t^{2^d})$, where d is an integer and $\tilde{\gamma} : (-\varepsilon', \varepsilon') \to \mathbb{R}^n$ is an injective analytic arc. Clearly we can consider $\lambda(0, \varepsilon)$ as one half-branch of the analytic set $\tilde{\gamma}(-\varepsilon', \varepsilon')$. Then the second branch of $\tilde{\gamma}(-\varepsilon', \varepsilon')$ is uniquely determined.

We recall now a basic definition from [**29**].

Definition 2.1. – Let M be a real analytic manifold. We say that a set $E \subset M$ is *arc-symmetric in M* if for every real analytic arc $\gamma(t) : (-1,1) \to M$, it has one of the following equivalent properties:

i) $$\operatorname{Int} \gamma^{-1}(E) \neq \emptyset \Longrightarrow \gamma((-1,1)) \subset E,$$

ii) $$\gamma((-1,0)) \subset E \Longrightarrow \gamma((-1,1)) \subset E.$$