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EFFECTIVE HEIGHT UPPER BOUNDS ON ALGEBRAIC TORI

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Abstract. – Let X be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ of the algebraic torus $\mathbf{G}_m^n$. We give an overview on what is known on upper bounds for the height when intersecting X with an algebraic subgroup of $\mathbf{G}_m^n$ that has dimension $n-\dim X$. Early general results in this directy were obtained by Bombieri-Zannier if X is a hypersurface and Bombieri-Masser-Zannier if X is a curve. Such height bounds are useful in the context of the Zilber-Pink Conjecture. The author proved a height bound for X of arbitrary dimension in 2009. In this paper we give an effective and explicit height upper bound.

***Résumé* (Bornes effectives pour la hauteur sur les tores algébriques).** – Soit X une sous-variété fermée, irréductible, définie sur $\overline{\mathbf{Q}}$, dans le tore algébrique $\mathbf{G}_m^n$. Nous donnons un survol des majorations de la hauteur d'un point de X, qui est contenu dans un sous-groupe algébrique de dimension $n - \dim X$. Les premiers résultats dans cette direction ont été obtenus par Bombieri-Zannier, dans le cas où X est une hypersurface, et par Bombieri-Masser-Zannier quand X est une courbe. Les majorations de ce type sont utiles pour résoudre quelques cas de la conjecture de Zilber-Pink. L'auteur a démontré une borne pour la hauteur, quand X est de dimension quelconque, en 2009. Nous explicitons ici une majoration effective.

0. Introduction

The main emphasis of this article is on attaining height upper bounds in the context of Boris Zilber and Richard Pink's conjectures. These open problems are often subsumed under the name of Zilber-Pink Conjecture. In the first section we briefly recount the development of the previous years while concentrating on situation in the algebraic torus. In Sections 2 through 10 we state and prove an effective and completely explicit version of the Bounded Height Conjecture, proved originally by

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the author [**31**]. We use, among other things, a recent result of Sturmfels and Tevelev from Tropical Geometry.

The first appendix gives a short overview of known results in the abelian case of abelian varieties. The second appendix contains a few height bounds in Shimura varieties.

1. A Brief Historical Overview in the Toric Setting

In 1999, Bombieri, Masser, and Zannier proved the following result for curves inside the algebraic torus $\mathbf{G}_m^n$. By coset we mean the translate of an algebraic subgroup of the ambient group variety.

***Theorem 1* (Bombieri, Masser, and Zannier [9]).** – *Let C be an irreducible algebraic curve inside $\mathbf{G}_m^n$ and defined over $\overline{\mathbf{Q}}$, an algebraic closure of $\mathbf{Q}$. Suppose that C is not contained in a proper coset, that is the translate of a proper algebraic subgroup. Then the height of points on C that are contained in a proper algebraic subgroup is bounded from above uniformly.*

The adjective uniformly refers to the fact that the bound for the height does not depend on the algebraic subgroup. It may and will depend on the curve C. Algebraic subgroups of the algebraic torus $\mathbf{G}_m^n$ are classified by subgroups of $\mathbf{Z}^n$, see Chapter 3.2 [**7**]. The difficulty in proving Bombieri, Masser, and Zannier's theorem arises from the fact that $\mathbf{G}_m^n$ has infinitely many connected algebraic subgroups of codimension 2.

Typically, C contains infinitely many points that are on a proper algebraic subgroup. After combining the height upper bound from Theorem 1 with a height lower bound due to Amoroso and David [**1**] in the spirit of Lehmer's problem, the authors obtained the following finiteness statement.

***Theorem 2* (Bombieri, Masser, and Zannier [9]).** – *Let C be as in Theorem 1. There are only finitely many points on C that are contained in an algebraic subgroup of codimension at least 2.*

This finiteness result is a special case, in dimension one, of a conjecture formulated by Boris Zilber [**76**] a few years later. His original conjecture, sometimes called the *Conjecture on Intersection with Tori* or short CIT, is a stronger version of the following conjecture.

Conjecture 1. – *Let X be an irreducible closed subvariety of $\mathbf{G}_m^n$ defined over $\mathbf{C}$. We suppose that X is not contained in a proper algebraic subgroup of $\mathbf{G}_m^n$. Then the set of points on X that are contained in the union of all algebraic subgroups of codimension at least $1 + \dim X$ is not Zariski dense in X.*

Points considered in this conjecture are in the intersection of a variety and a subgroup whose codimension is too small to expect non-empty intersection. Such intersections are deemed *unlikely* but are certainly not impossible. In fact, Zilber's full conjecture governs also subvarieties of X that are contained in a surprisingly small algebraic subgroup. It is also expected to hold for the wider class of semi-abelian varieties, which includes algebraic tori and abelian varieties. Finally, Pink [**50, 49**] has proposed a related conjecture for mixed Shimura varieties which implies the André-Oort Conjecture. It also contains Conjecture 1 above; many conjectures in this setting are sometimes called Zilber-Pink Conjecture. See Ullmo's contribution [**67**] to this volume.

Example. – We exemplify the situation on the example of a line in $\mathbf{G}_m^3$. The line is parametrized by

$$(\alpha t+\alpha', \beta t+\beta', \gamma t+\gamma') \tag{1.1}$$

with $\alpha,\beta,\gamma,\alpha',\beta',\gamma' \in \overline{\mathbf{Q}}$ fixed. To guarantee that our line is not contained in a proper coset of $\mathbf{G}_m^3$ we ask that

$$\alpha\beta\gamma \neq 0 \quad \text{and} \quad \frac{\alpha'}{\alpha}, \frac{\beta'}{\beta}, \frac{\gamma'}{\gamma} \text{ are pair-wise distinct.}$$

Theorem 1 implies that there is a constant B with the following property. Suppose $t \in \overline{\mathbf{Q}}$ such that no coordinate of (1.1) vanishes and such that there exist $(a,b,c) \in \mathbf{Z}^3$ with

$$(\alpha t+\alpha')^a(\beta t+\beta')^b(\gamma t+\gamma')^c = 1 \tag{1.2}$$

then the height of t is at most B.

The exponent vector (a,b,c) is allowed to vary in (1.2). It determines a proper algebraic subgroup of $\mathbf{G}_m^3$. Conversely, any such subgroup arises in this way.

In order to obtain finiteness of the set of t as in the second result of Bombieri, Masser, and Zannier we must impose a second condition. More precisely, there are only finitely many t with (1.2) such that there is $(a',b',c') \in \mathbf{Z}^3$ linearly independent of (a,b,c) with

$$(\alpha t+\alpha')^{a'}(\beta t+\beta')^{b'}(\gamma t+\gamma')^{c'} = 1.$$

The two monomial equations determined by (a,b,c) and (a',b',c') define an algebraic subgroup of $\mathbf{G}_m^3$ of codimension 2.

Bombieri, Masser, and Zannier [**9**] remarked that any curve contained in a proper coset invariably leads to unbounded height. So this condition cannot be dropped from their theorem. On the other hand, it remained unclear if the restriction was necessary in order to achieve finiteness in the situation of unlikely intersections. The authors posed the following question.

Question 1. – Suppose $C \subset \mathbf{G}_m^n$ is an irreducible algebraic curve defined over $\overline{\mathbf{Q}}$ that is not contained in a proper algebraic subgroup. Is the set of points on C that are contained in an algebraic subgroup of codimension 2 finite?

First progress was made in 2006 when the same group of authors obtained a partial answer by making a detour to surfaces if $n = 5$.

Theorem 3 **(Bombieri, Masser, and Zannier [10]).** – *Suppose $n \leq 5$. Let C be an irreducible algebraic curve inside $\mathbf{G}_m^n$ and defined over $\overline{\mathbf{Q}}$. Suppose that C is not contained in a proper algebraic subgroup. There are only finitely many points on C that are contained in an algebraic subgroup of codimension at least* 2.

In the most interesting case $n = 5$ they constructed an algebraic surface $S \subset \mathbf{G}_m^3$ derived from the curve. In order to answer their question for C, a bounded height result akin to Theorem 1 was needed for points on S lying in an algebraic subgroup of codimension $\dim S = 2$, i.e., of dimension 1. Luckily, height bounds on the intersection of a fixed variety with varying one dimensional algebraic subgroups were known to follow from earlier work of Bombieri and Zannier.

Theorem 4 **(Bombieri and Zannier [74]).** – *Suppose $X \subset \mathbf{G}_m^n$ is an irreducible algebraic subvariety defined over $\overline{\mathbf{Q}}$. Let X° be the complement in X of the union of all cosets of positive dimension that are contained in X. Then the height of points on X° that are contained in an algebraic subgroup of dimension at most* 1 *is uniformly bounded.*

The construction for S used by the three authors works for $n > 5$ too and always yields a surface in some $\mathbf{G}_m^n$. However, recovering (3) for $n > 5$ would require a height bound on the intersection of S with algebraic subgroups of dimension greater than one. Thus Theorem 4 is no longer applicable.

One interesting aspect of Theorem 3 is that its proof is effective, provided an effective height bound in Theorem 4 is available. The author [**29**] proved a generalization of Theorem 4 that is effective and completely explicit. The height bound for the points considered in Theorem 4 depended on n and the degree and height of X.

In 2008, and using a different approach, Maurin gave a positive answer to Question 1 for all n.

Theorem 5 **(Maurin [40]).** – *Let C be an irreducible algebraic curve inside $\mathbf{G}_m^n$ and defined over $\overline{\mathbf{Q}}$. Suppose that C is not contained in a proper algebraic subgroup. There are only finitely many points on C that are contained in an algebraic subgroup of codimension at least* 2.

Maurin first proves that the height is bounded from above. But his method differed substantially from the one used in the proofs of Theorems 1 and 3. It relied on generalization of Vojta's inequality by Rémond [**53**] which accounts for the varying algebraic subgroups. The original Vojta inequality appeared prominently in a proof of Faltings' Theorem, the Mordell Conjecture.

Maurin's Theorem holds for curves defined over $\overline{\mathbf{Q}}$. But its conclusion makes no reference to algebraic numbers. Having it at their disposal Bombieri, Masser, and Zannier applied specialization techniques to obtain the following result.