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SPECIAL POINTS AND INTERSECTIONS IN ABELIAN AND SHIMURA VARIETIES

by

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Abstract. – In this paper we give an overview of the proofs of the Manin-Mumford and the André-Oort (assuming the Generalised Riemann Hypothesis) conjectures based on a combination of Galois-theoretic, Geometric and Ergodic techniques.

Résumé **(Points spéciaux et intersections dans les variétés abéliennes et les variétés de Shimura)**

Dans ce texte nous présentons les idées centrales des preuves des conjectures de Manin-Mumford et d'André-Oort (admettant l'Hypothèse de Riemann Généralisée) basées sur la combinaison des techniques galoisiennes, géométriques et ergodiques.

1. Introduction

These notes are expanded versions of the lectures given at the 'États de la recherche' conference held at CIRM, Luminy in May 2011. The conference was devoted to the recent developments on the André-Oort and Zilber-Pink conjectures. My lectures were given jointly with Emmanuel Ullmo and our aim was to present, in a reasonable amount of detail, the proof of the André-Oort conjecture (assuming the Generalized Riemann Hypothesis or GRH for short) which combines Galois-theoretic and Ergodic techniques. These notes partly rely on Ullmo's notes (see [**25**]) in this volume which provide the necessary background.

Furthermore, Ullmo's text [**26**] gives a good introduction to the abstract formulation of the Zilber-Pink conjectures and we use his terminology and notations.

The main goal of these notes is to give an overview of the strategies of the proofs. For technical details we refer to the original papers [**28**] and [**14**].

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Statements such as the André-Oort and the Zilber-Pink conjectures can be formulated most generally in the context of mixed Shimura varieties. For this we refer to Ullmo's text [**26**].

In these notes, we focus on the cases of Abelian and pure Shimura varieties. We start with some abstract preliminaries on Equidistribution, in particular, we make precise what we mean by equidistribution in our context.

We then proceed to explain the implementation of our strategy in the case of Abelian varieties (the Manin-Mumford conjecture). The proof follows the same lines as the Shimura case but each ingredient is much easier to formulate and to prove, and the assumption of the GRH is not needed in this case.

Finally, we sketch the proof of the André-Oort conjecture assuming the GRH using a strategy which combines Galois-theoretic techniques with Geometric techniques and Equidistribution. We would like to point out here that a proof under GRH following the same lines, but avoiding the equidistribution altogether (and using some group-theoretic considerations instead) was carried out by Daw (see [**4**]).

Quite recently, Jonathan Pila and Umberto Zannier came up with a new approach to the Manin-Mumford and André-Oort conjectures which uses ideas and results from the theory of o-minimality and gives a hope to prove the André-Oort conjecture unconditionally. It yielded a number of unconditional results on the André-Oort conjecture (see [**15**]. [**16**], [**17**] and [**6**]) culminating in the unconditional proof of the André-Oort conjecture for all Shimura varieties of Abelian type (see [**23**] and references therein). It should also be pointed out, that a new proof of the André-Oort conjecture under GRH using the Pila-Zannier strategy is now available by combining lower bounds for Galois orbits of special points ([**29**] or [**22**]) under GRH, with the hyperbolic Ax-Lindemann-Weierstrass conjecture proven in [**13**] and upper bounds on heights of pre-special points proven in [**5**] by Daw and Orr. An overview of Pila-Zannier's approach as well as necessary background on o-minimality can be found in Scanlon's notes [**20**] in this volume. Another good reference for this approach is Daw's survey [**3**] (this survey is closer in spirit to these notes).

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2. Preliminaries on Equidistribution

In this section we consider a quasi-projective algebraic variety X over $\mathbb{C}$ endowed with a special structure as defined in **[26]** and assume that the set of special subvarieties $\Sigma(X)$ is countable.

We furthermore assume that X is endowed with a canonical probability measure that we call μ_X and that each special subvariety Z is endowed with a canonical probability measure μ_Z with support Z. This will indeed be true in the cases we are going to consider. In particular, this is true for mixed Shimura varieties.

Let $P(X)$ be the set of Borel probability measures on X and let $C(X)$ be the set of *bounded* continuous functions on X.

Definition 2.1. – *We say that a sequence (μ_n) of elements of $P(X)$ is weakly convergent to $\mu \in P(X)$ if for all $f \in C(X)$*

$$\mu_n(f) = \int_X f d\mu_n \longrightarrow \mu(f) = \int_X f d\mu \text{ as } n \longrightarrow \infty.$$

We write

$$\mu_n \longrightarrow \mu.$$

The usual definition of equidistribution of a subsequence of $P(X)$ is as follows.

Definition 2.2. – *A sequence of measures (μ_n) in $P(X)$ is said to equidistribute if μ_n converges to μ_X.*

In the case of measures attached to special subvarieties we will take a slightly different definition of Equidistribution which is as follows.

***Definition 2.3* (Equidistribution)**. – *We say that a sequence of special subvarieties (Z_n) is* equidistributed *if there exists a special subvariety Z such that after possibly extracting a subsequence the following conditions hold:*

1. $\mu_{Z_n} \longrightarrow \mu_Z$ *and*
2. $Z_n \subset Z$ *for all n large enough.*

We will need the following definition.

Definition 2.4. – *Let X be as before and Z a special subvariety of X. A sequence of special subvarieties (Z_n) of Z is called* strict *(relative to Z) if any* proper *special subvariety of Z contains only finitely many Z_ns.*

We have the following.

Proposition 2.5. – *Suppose that (Z_n) and all its subsequences are equidistributed in the sense of Definition 2.3. Then components of the Zariski closure of $\bigcup_n Z_n$ are special.*

Proof. – Let Y be component of the Zariski closure of $\bigcup_n Z_n$. Replace (Z_n) by the subsequence of Z_ns contained in Y. Let Z_Y be the smallest special subvariety of X containing Y (such a Z_Y exists by definition of a special structure). The set of proper special subvarieties of Z_Y is countable and we number them as (Z'_n).

For any $k \geq 1$, let Z_{n_k} be an element of (Z_n) not contained in $\bigcup_{i=1}^k Z'_k$. Such a choice is clearly possible by the Zariski density and by construction (Z_{n_k}) is strict relative to Z_Y. By the equidistribution property there exists a special subvariety $Z \subset Y$ containing infinitely many Z_{n_k}s which implies that $Z = Y = Z_Y$ and hence Y is special. □

In fact, a stronger statement than the (abstract) André-Oort conjecture (in cases where it makes sense) is the so-called *Equidistribution conjecture* which is the following.

Assume that X is defined over $\overline{\mathbb{Q}}$ and let F be a number field of definition of X. Assume that all special subvarieties are also defined over $\overline{\mathbb{Q}}$. To any $x \in X(\overline{\mathbb{Q}})$, one attaches the following probability measure

$$\Delta_x = \frac{1}{[F(x) : F]} \sum_{y \in \mathrm{Gal}(\overline{F}/F) \cdot x} \delta_y$$

where δ_y denotes the Dirac measure with support in $\{y\}$.

***Conjecture 2.6* (Equidistribution conjecture).** – *Let x_n be a* strict *(with respect to X) sequence of special points in X. Then*

$$\Delta_{x_n} \longrightarrow \mu_X.$$

There are few cases of Shimura varieties for which this conjecture is known. It is only known for modular curves in full generality (Duke, [**9**]). Zhang (see [**30**]) proved the Equidistribution conjecture for some quaternionic Shimura varieties but only for sequences satisfying additional assumptions.

We refer to Ullmo's text [**24**] for a thorough account of the Equidistribution approach to the Manin-Mumford and the André-Oort type of questions.

The Equidistribution conjecture is one of the directions for future developments. One has to prove two things: one is the Equidistribution of *toric* orbits and the other is to derive from it the Equidistribution of Galois orbits. There is some recent work on the Equidistribution of toric orbits by Eiseindler, Linderstrauss, Venkatesh and Ph. Michel.