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KNOTS, FLOWS, AND FLUIDS

by

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Abstract. — This paper is organized along three directions which are strongly interconnected. The first one is a review of topological invariants associated with flows in 3 dimensional manifolds. These invariants are related to averaged asymptotic linking properties of orbits. The second one is the study of the space of configurations of an incompressible fluid on an oriented manifold. This space can be parametrized by the group of diffeomorphisms isotopic to the identity which preserve a given volume form. This group can be equipped with a natural right invariant metric that we analyse both from the global point of view (diameter of the group) and the geodesic one. Extensions to the case of groups of diffeomorphisms preserving a given symplectic form are also reviewed. A particular attention is paid to the case when the dimension is 2 where both points of view coincide. In the third part, we study the structure of the group of diffeomorphisms on a compact oriented surface which are isotopic to the identity and preserve a given area form. According to the genus of the surface, we describe the subgroup generated by the commutators and show that on this subgroup the commutator length is unbounded, a result obtained in collaboration with Étienne Ghys.

***Résumé* (Nœuds, Flots, et Fluides).** — Cet article s'organise autour de trois directions fortement corrélées. La première donne une description de divers invariants topologiques associés à un flot préservant le volume sur une variété de dimension 3. Ces invariants sont reliés aux propriétés d'enlacement moyen asymptotique des orbites. La deuxième direction concerne l'étude de l'espace des configurations d'un fluide incompressible sur une variété orientée. Cet espace peut être paramétré par le groupe des difféomorphismes isotopes à l'identité, préservant une forme volume. Ce groupe est équipé d'une métrique naturelle invariante à droite que l'on étudie tant du point de vue global (diamètre du groupe) que du point de vue géodésique. Le cas des groupes de difféomorphismes préservant une forme symplectique est également abordé en suivant une même approche. Une attention toute particulière est portée au cas de la dimension 2 où les deux points de vue se rejoignent. Dans une troisième direction, on étudie la structure du groupe des difféomorphismes isotopes à l'identité et préservant une forme d'aire sur les surfaces compactes orientées. Selon le genre de la surface, on décrit le sous-groupe des commutateurs et on montre sur ce sous-groupe que la fonction longueur des commutateurs n'est pas bornée, un résultat obtenu en collaboration avec Étienne Ghys.

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1. Introduction

1.1. Gone with the wind. — In the year 1858, Herman Ludwig Ferdinand von Helmholtz published in *Crelle's Journal* [**32**] a deep and pioneering paper on vortex motions, entitled *"Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen"*.

This work concerns the motion of an ideal fluid moving in a domain $\mathcal{D}^3$ of the 3-dimensional Euclidean space $\mathbb{R}^3$. The basic observation made by Helmoltz is that, at a given time t, the infinitesimal motion of a volume element of the fluid can be decomposed in the sum of a translation, a pure straining motion, and a rigid body rotation. The axis of this infinitesimal rigid body rotation at each point x in $\mathcal{D}^3$ defines a line field in $\mathcal{D}^3$, whose integral curves are called *vortex lines*.

In a more mathematical language, if we denote by $u(x,t)$ the velocity of a fluid particle located at x in $\mathcal{D}^3$, at time t, the infinitesimal variation of the velocity of the fluid at x reads:

$$d_x u = E + \Omega$$

where E is a symmetric tensor which represents the infinitesimal pure straining motion, and Ω is an anti-symmetric tensor which represents the infinitesimal rotation. The *vorticity* vector $\omega(x,t) = \operatorname{curl} u(x,t)$ at x in $\mathcal{D}^3$ and at time t, is parallel to the axis of the infinitesimal rotation $\Omega(x,t)$. At each time t, vortex lines are thus orbits of the vorticity vector field. Notice that from its very definition, the vorticity vector field is *solenoidal*, i.e. it is divergence free or equivalently its flow preserves the standard volume form in $\mathcal{D}^3$.

Let D be the closed ball with radius 1 centered at 0 in $\mathbb{R}^2$, ∂D its border and I the closed interval $[0,1]$. Consider an embedding $\mathcal{E} : D \times I \to \mathcal{D}^3$ such that:

– for all c in I, the image of $D \times \{c\}$ is a disk transverse to the vortex lines;
– for all x in D, the image of $\{x\} \times I$ belongs to a vortex line.

The image of the cylinder $\partial D \times I$ by such an embedding is called a *vortex tube*.

Let $\mathcal{C}$ be a closed oriented curve on $\partial D \times I$ homotopic to $\partial D \times \{c\}$, where ∂D is equipped with its standard orientation and c is any point in I. The *strength* of the vortex tube $\mathcal{E}(\partial D \times I)$ is the circulation along the curve $\mathcal{E}(\mathcal{C})$:

$$\oint_{\mathcal{E}(\mathcal{C})} u \,.\, ds$$

As noticed by Helmoltz (it can be seen as a simple application of Stokes theorem) the strength of a vortex tube does not depend on the choice of the curve $\mathcal{C}$ and is equal to the flow of the vorticity field through the embedded disk $\mathcal{E}(D \times \{c\})$, for any c in I.

Helmoltz aim was to describe the time evolution of vortex lines and tubes along a fluid motion. In the case of an ideal barotropic fluid under the action of conservative external body forces (we will recall in a while what this means from our mathematical point of view), he derived the following three laws:

Under time evolution,

(I) *Fluid particles originally free of vorticity remain free of vorticity;*
(II) *Vortex lines and tubes remain vortex lines and tubes;*

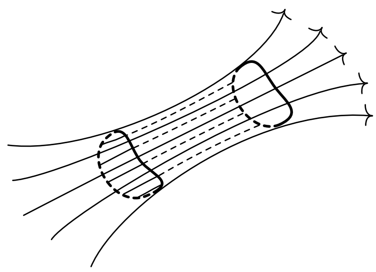

FIGURE 1. A vortex tube

(III) *The strength of a vortex tube remains unchanged.*

Observing the time evolution equations associated with the fluid motion gives an insight of the proof of these laws. Let ρ be the density of the fluid, u its velocity, and F the external force per unit mass. When the fluid is ideal, Euler equations describe conservation of mass and momentum:

$$\frac{\partial\rho}{\partial t} + u\,.\,\nabla\rho + \rho\operatorname{div} u = 0 \tag{1}$$

and

$$\frac{\partial u}{\partial t} + (u\,.\,\nabla)u = -\Big(\frac{1}{\rho}\Big)\nabla p + F, \tag{2}$$

where p is the pressure (the cohesion force of the fluid).

Applying curl operator to Equation (2) yields the general time evolution equation for vorticity:

$$\frac{\partial\omega}{\partial t} + u\,.\,\nabla\omega = \omega\,.\,\nabla u - \omega\operatorname{div} u + \Big(\frac{1}{\rho^2}\Big)\nabla\rho\wedge\nabla p + \operatorname{curl} F. \tag{3}$$

The fluid is said barotropic when the surfaces of constant pressure are parallel to the surfaces of constant density: in this case the term $\nabla\rho\wedge\nabla p$ vanishes. Whenever the external body force is conservative the term $\operatorname{curl} F$ also vanishes and Equation (3) reduces to:

$$\frac{\partial\omega}{\partial t} + (u\,.\,\nabla)\omega = (\omega\,.\,\nabla)u - \omega\operatorname{div} u. \tag{4}$$

Combining with Equation (1) we get:

$$\frac{\partial}{\partial t}\frac{\omega}{\rho} + (u\,.\,\nabla)\frac{\omega}{\rho} - \Big(\frac{\omega}{\rho}\,.\,\nabla\Big)u = 0, \tag{5}$$

which also reads:

$$\frac{\partial}{\partial t}\frac{\omega}{\rho} + \Big[u, \frac{\omega}{\rho}\Big] = 0, \tag{6}$$

where $[.,.]$ stands for the Lie bracket.

Equation (6) has a very nice geometrical meaning: it says that the pondered vorticity vector field ω/ρ is *frozen* in the velocity vector field. More precisely, given a time t and a positive real number τ, consider the map ${\Phi^\tau}_t : \mathcal{D}^3 \to \mathcal{D}^3$ which, to any position

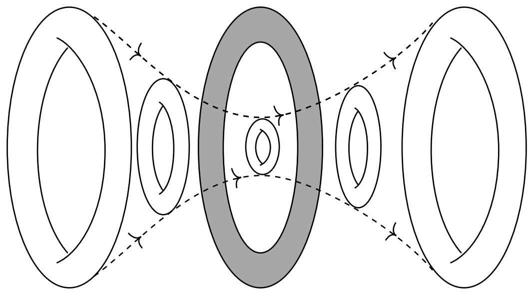

Figure 2. Smoke rings parade

x in $\mathcal{D}^3$, associates the position at time $t+\tau$ of the particle that was in x at time t. Assume furthermore that this map is smooth, invertible and with a smooth inverse. Equation (6) implies that ${\Phi^\tau}_t$ realizes a conjugacy between the vorticity fields at time t and time $t+\tau$:

$$\frac{\omega}{\rho}(.,t+\tau) = {{\Phi^\tau}_t}^\star\Big(\frac{\omega}{\rho}(.,t)\Big)$$

or more explicitly

$$\frac{\omega}{\rho}({\Phi^\tau}_t(x), t+\tau) = d_x{\Phi^\tau}_t\Big(\frac{\omega}{\rho}(x,t)\Big) \quad \forall x \in \mathcal{D}^3.$$

Laws $(I), (II)$ and (III) turn to be direct consequences of this conjugacy relation.

A particularly elegant illustration of Helmoltz laws concerns vortex rings. A *vortex ring* is an embedded 2-torus in $\mathcal{D}^3$ tangent to the vorticity vector field and which is the boundary of an embedded solid torus. Under time evolution vortex rings remain vortex rings even if deformed, and particles which are inside a vortex ring at some time t cannot escape from the ring as time goes.

In his paper, Helmoltz went further than the topological approach we described above and analyzed time evolution of a pair of neighbor vortex rings in a fluid. His conclusions read as follows:

> *"If they both have the same direction of rotation they will proceed in the same sense, and the ring in front will enlarge itself and move slower, while the second one will shrink and move faster, if the velocities of translation are not too different, the second will finally reach the first and pass through it. Then the same game will be repeated with the other ring, so the ring will pass alternately one through the other."*

This profound paper by Helmoltz was not too paid attention to when published. It had to wait until the English translation by Peter Guthrie Tait in 1867. On top of his translation work, Tait imagined and realized a physical experiment showing the exactness of the theoretical conclusions by Helmoltz on the dynamics of pairs of vortex rings.

Tait's idea was to encapsulate smoke in a vortex ring for visualization. The smoke rings persist a long time after they had been created (as long as dissipative effects in

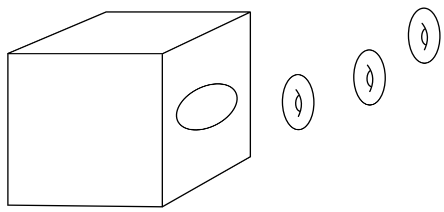

FIGURE 3. Tait's machine

the atmospheric air motion do not destroy the pertinence of Euler equations). Let us follow the careful description by Thomson [**52**] of Tait's machine:

> *"A large rectangular box, open at one side, has a circular hole of 6 or 8 inches diameter cut in the opposite side. A common rough packing-box of 2 feet cube, or thereabout, will answer the purpose very well. The open side of the box is closed by a stout towel or piece of cloth, or by a sheet of india-rubber stretched across it. A blow on this flexible side causes a circular vortex ring to shoot out from the hole on the other side. The vortex rings thus generated are visible if the box is filled with smoke. One of the most convenient ways of doing this is to use two retorts with their necks thrust into holes made for the purpose in one of the sides of the box. A small quantity of muriatic acid is put into one of these retorts, and of strong liquid ammonia into the other. By a spirit-lamp applied from time to time to one or the other of these retorts, a thick cloud of sal-ammoniac is readily maintained in the inside of the box. A curious and interesting experiment may be made with two boxes thus arranged, and placed either side by side close to one another or facing one another so as to project smoke-rings meeting from opposite directions – or in various relative positions, so as to give smoke-rings proceeding in paths inclined to one another at any angle, and passing one another at various distances. An interesting variation of the experiment may be made by using clear air without smoke in one of the boxes. The invisible vortex rings projected from it render their existence startingly sensible when they come near any of the smoke-rings proceeding from the other box."*

Thomson soon realized the analogy between fluid dynamics and electromagnetism(1). Recall that at that time, Maxwell equations were dominating physics, electromagnetic waves were thought to be vibrations in the *"luminiferous ether"*, and atoms were still mysterious unidentified objects. This led Thomson to propose a theory where atoms were knotted vortices rings in the ether. Thomson's ideas turned to be completely wrong but his enthusiasm had a very deep impact on the development of mathematics. He encouraged Tait to settle down the basis of what is now called *knot theory* [**49, 50, 51**]. Tait started trying to list knots according to their numbers of double points when projected to a plane in a most efficient way and

(1)See the second paragraph in this Section.