CATEGORICAL SEMANTICS OF LINEAR LOGIC

Paul-André Melliès

Panoramas et Synthèses

Numéro 27

2009

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique

Panoramas & Synthèses
27, 2009, p. 1–197

CATEGORICAL SEMANTICS OF LINEAR LOGIC

by

Paul-André Melliès

Abstract. – Proof theory is the result of a short and tumultuous history, developed on the periphery of mainstream mathematics. Hence, its language remains often idiosyncratic: sequent calculus, cut-elimination, subformula property, etc. This survey is designed to guide the novice reader and the itinerant mathematician along a smooth and consistent path, investigating the symbolic mechanisms of cut-elimination, and their algebraic transcription as coherence diagrams in categories with structure. This spiritual journey at the meeting point of linguistic and algebra is demanding at times, but a rewarding experience: to date, no language (either formal or informal) has been studied by mathematicians as thoroughly as the language of proofs.

***Résumé* (Sémantique catégorielle de la logique linéaire).** – La théorie de la démonstration est issue d'une histoire courte et tumultueuse, construite en marge des mathématiques traditionnelles. Aussi, son langage reste souvent idiosyncratique : calcul des séquents, élimination des coupures, propriété de la sous-formule, etc. Dans cet article, nous avons voulu guider le lecteur à travers la thématique, en lui traçant un chemin progressif et raisonné, qui part des mécanismes symboliques de l'élimination des coupures, pour aboutir à leur transcription algébrique en diagrammes de cohérence dans les catégories monoïdales. Cette promenade spirituelle au point de convergence de l'algèbre et de la linguistique est ardue parfois, mais aussi pleine d'attraits : car à ce jour, aucune langue (formelle ou informelle) n'a été autant étudiée par les mathématiciens que la langue des démonstrations logiques.

Proof theory is the result of a short and tumultuous history, developed on the periphery of mainstream mathematics. Hence, its language is often idiosyncratic: sequent calculus, cut-elimination, subformula property, etc. This survey is designed to

2010 Mathematics Subject Classification. – 03-02, 03B05, 03B10, 03B15, 03B20, 03B22, 03B40, 03B45, 03B47, 03B70, 03F03, 03F05, 03F07, 03F52, 03G30, 18A15, 18A40, 18C20, 18C50, 18D05, 18D10, 18D15, 18D25, 91A05, 91A20, 91A40, 91A80.
Key words and phrases. – Proof theory, sequent calculus, cut-elimination, categorical semantics, linear logic, monoidal categories, $*$-autonomous categories, linearly distributive categories, dialogue categories, string diagrams, functorial boxes, 2-categories, monoidal functors, monoidal adjunctions, coherence spaces, game semantics.

guide the novice reader and the itinerant mathematician along a smooth and consistent path, investigating the symbolic mechanisms of cut-elimination, and their algebraic transcription as coherence diagrams in categories with structure. This spiritual journey at the meeting point of linguistic and algebra is demanding at times, but also pleasantly rewarding: to date, no language (either formal or informal) has been studied by mathematicians as thoroughly as the language of proofs.

We start the survey by a short introduction to proof theory (Chapter 1) followed by an informal explanation of the principles of denotational semantics (Chapter 2) which we understand as a representation theory for proofs – generating algebraic invariants modulo cut-elimination. After describing in full detail the cut-elimination procedure of linear logic (Chapter 3), we explain how to transcribe it into the language of categories with structure. We review three alternative formulations of $*$-autonomous category, or monoidal category with classical duality (Chapter 4). Then, after giving a 2-categorical account of lax and oplax monoidal adjunctions (Chapter 5) and recalling the notions of monoids and monads (Chapter 6) we relate four different categorical axiomatizations of propositional linear logic appearing in the literature (Chapter 7). We conclude the survey by describing two concrete models of linear logic, based on coherence spaces and sequential games (Chapter 8) and by discussing a series of future research directions (Chapter 9).

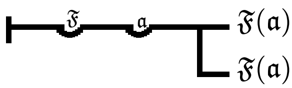
1. Proof theory: a short introduction

From vernacular proofs to formal proofs: Gottlob Frege. – By nature and taste, the mathematician studies properties of specific mathematical objects, like rings, manifolds, operator algebras, etc. This practice involves a high familiarity with proofs, and with their elaboration. Hence, building a proof is frequently seen as an art, or at least as a craft, among mathematicians. Any chair is fine to sit on, but some chairs are more elegant than others. Similarly, the same theorem may be established by beautiful or by ugly means. But the experienced mathematician will always look for an elegant proof.

In his daily work, the mathematician thinks of a proof as a rational argument exchanged on a blackboard, or exposed in a book – without further inquiry. The proof is seen as a vehicle of thought, not as an object of formal investigation. In that respect, the logician interested in *proof theory* is a peculiar kind of mathematician: one who investigates *inside the language of mathematics* the linguistic event of convincing someone else, or oneself, by a mathematical argument.

Proof theory really started in 1879, when Gottlob Frege, a young lecturer at the University of Iena, published a booklet of eighty-eight pages, and one hundred thirty-three formulas [**33**]. In this short monograph, Frege introduced the first mathematical notation for proofs, which he called *Begriffschrift* in German – a neologism translated today as *ideography* or *concept script*. In his introduction, Frege compares this ideography to a microscope which translates *vernacular proofs* exchanged between mathematicians into *formal proofs* which may be studied like any other mathematical object.

In this formal language invented by Frege, proofs are written in two stages. First, a formula is represented as 2-dimensional graphical structures: for instance, the syntactic tree



is a graphical notation for the formula written

$$\forall \mathfrak{F}.\ \forall \mathfrak{a}.\ \mathfrak{F}(\mathfrak{a}) \Rightarrow \mathfrak{F}(\mathfrak{a})$$

in our contemporary notation – where the first-order variable $\mathfrak{a}$ and the second-order variable $\mathfrak{F}$ are quantified universally. Then, a proof is represented as a sequence of such formulas, constructed incrementally according to a series of *derivation rules*, or logical principles. It is remarkable that Frege introduced this language of proofs, and formulated in it the first theory of quantification.

Looking for Foundations: David Hilbert. – Despite his extraordinary insight and formal creativity, Gottlob Frege remained largely unnoticed by the mathematical community of his time. Much to Frege's sorrow, most of his articles were rejected by mainstream mathematical journals. In fact, the few contemporary logicians who read the ideography generally confused his work with George Boole's algebraic account of logic. In a typical review, a prominent German logician of the time describes the 2-dimensional notation as "a monstrous waste of space" which "indulges in the Japanese custom of writing vertically". Confronted to negative reactions of that kind, Frege generally ended up rewriting his mathematical articles in a condensed and non technical form, for publication in local philosophical journals.

Fortunately, the ideography was saved from oblivion at the turn of the century, thanks to Bertrand Russell – whose curiosity in Frege's work was initially aroused by a review by Giuseppe Peano, written in Italian [**77**]. At about the same time, David Hilbert, who was already famous for his work in algebra, got also interested in Gottlob Frege's ideography. On that point, it is significant that David Hilbert raised a purely proof-theoretic problem in his famous communication of twenty-three open problems at the International Congress of Mathematicians in Paris, exposed as early as 1900. The second problem of the list consists indeed of showing that arithmetic is consistent, that is, without contradiction.

David Hilbert further develops this idea in his monograph on the Infinite, written 25 years later [**47**]. He explains there that he hopes to establish, by purely *finite* combinatorial arguments on formal proofs, that there exists no contradiction in mathematics — in particular no contradiction in arguments involving *infinite* objects in arithmetic and analysis. This finitist program was certainly influenced by his successful work in algebraic geometry, which is also based on the finitist principle of reducing the infinite to the finite. This idea may also have been influenced by discussions with

Frege. However, Kurt Gödel established a few years later, with his incompleteness theorem (1931) that Hilbert's program was a hopeless dream: consistency of arithmetics cannot be established by purely arithmetical arguments.

Consistency of Arithmetics: Gerhard Gentzen. – Hilbert's dream was fruitful nonetheless: Gerhard Gentzen (who was originally a student of Hermann Weyl) established the consistency of arithmetics in 1936, by a purely combinatorial argument on the structure of arithmetic proofs. This result seems to contradict the fact just mentioned about Gödel's incompleteness theorem, that no proof of consistency of arithmetic can be performed inside arithmetic. The point is that Gentzen used in his argument a transfinite induction up to Cantor's ordinal ε_0 – and this part of the reasoning lies outside arithmetics. Recall that the ordinal ε_0 is the first ordinal in Cantor's epsilon hierarchy: it is defined as the smallest ordinal which cannot be described starting from zero, and using addition, multiplication and exponentiation of ordinals to the base ω.

Like many mathematicians and philosophers of his time, Gerhard Gentzen was fascinated by the idea of providing safe foundations (*Grundlagen* in German) for science and knowledge. By proving consistency of arithmetic, Gentzen hoped to secure this part of mathematics from the kind of antinomies or paradoxes discovered around 1900 in Set Theory by Cesare Burali-Forti, Georg Cantor, and Bertrand Russell. Today, this purely foundational motivation does not seem as relevant as it was in the early 1930s. Most mathematicians believe that reasoning by finite induction on natural numbers is fine, and does not lead to contradiction in arithmetics. Besides, it seems extravagant to convince the remaining skeptics that finite induction is safe, by exhibiting Gentzen's argument based on transfinite induction...

The sequent calculus. – For that reason, Gentzen's work on consistency could have been forgotten along the years, and reduced in the end to a dusty trinket displayed in a cabinet of mathematical curiosity. Quite fortunately, the contrary happened. Gentzen's work is regarded today as one of the most important and influential contributions ever made to logic and proof theory. However, this contemporary evaluation of his work requires to reverse the traditional perspective: what matters today is not the consistency result in itself, but rather the method invented by Gerhard Gentzen in order to establish this result.

This methodology is based on a formal innovation: the *sequent calculus* and a fundamental discovery: the *cut-elimination theorem.* Together, this calculus and theorem offer an elegant and flexible framework to formalize proofs — either in classical or in intuitionistic logic, as Gentzen advocates in his original work, or in more recent logical systems, unknown at the time, like linear logic. The framework improves in many ways the formal proof systems designed previously by Gottlob Frege, Bertrand Russell, and David Hilbert. Since the whole survey is based on this particular formulation of logic, we find it useful to explain below the cardinal principles underlying the sequent calculus and its cut-elimination procedure.