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HIERARCHY OF MODELS AND NUMERICAL APPROXIMATION

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GREY RADIATIVE HYDRODYNAMICS — HIERARCHY OF MODELS AND NUMERICAL APPROXIMATION

by

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Abstract. – We address grey models for the coupling of radiation and hydrodynamics, having in mind the derivation of eulerian numerical methods. It raises many mathematical and numerical difficulties. In this text we focus on some of them which are: the grey non equilibrium diffusion model, moments models and asymptotic preserving numerical methods.

Résumé **(Hydrodynamique radiative grise ; hiérarchie de modèles et approximation numérique)**
Nous considérons les modèles dits gris pour le couplage du rayonnement et de l'hydrodynamique, dans l'objectif de construire des méthodes de résolution numériques eulériennes. Cela pose de nombreuses difficultés mathématiques et numériques. Nous en considérons quelques unes en portant l'accent sur les modèles de diffusion hors équilibre, les modèles aux moments et les schémas « asymptotic preserving ».

1. Introduction

This research was motivated by numerical studies for basic models arising in Inertial Confinement Fusion (ICF), see recent works [**8, 9, 10**].

We address the coupling of radiation and hydrodynamics, in view of the construction robust schemes for the numerical solutions of such problems. Following Lowrie, Morel and Hittinger in [**27, 28**] we consider that numerical progress should be possible using Eulerian conservative formulation of the model. But it raises many difficulties: many models are written in the comobile or Lagrangian reference frame which moves with the fluid, see [**3, 29, 30**], [**35**], [**12**], [**33**], [**38**] and references therein. For mathematical aspects of the radiative transfer equation and related issues see for instance [**4**]. Actually one can distinguish at least three approaches: a purely Lagrangian approach where everything is calculated in the moving reference frame; a comobile approach

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where some quantities are calculated in Eulerian reference frame and others are calculated in the moving reference frame (see for example [**33**]); the last approach is purely Eulerian, everything is calculated in the Eulerian reference frame. At the theoretical level, we owe a lot to D. Mihalas (we refer the reader to the modern presentation [**22**]) and Lowrie, Morel and Hittinger [**28**]; the only originality of our presentation lays in the derivation of the non equilibrium diffusion limit from the characterization in the lab's frame (16) of the anisotropy in the comobile frame of the scattering of photons. Then we discuss carefully the compatibility of the M^1 Levermore moment model [**24**] in this context. Finally we explain how to rewrite the M^1 model as an almost standard gas dynamics system using new unknowns.

In section 2 we detail an Eulerian based analysis of grey asymptotic regimes and grey models for radiative hydrodynamics: all equations are written in the lab's frame, even if the source is compatible with the comobile frame. We start with the transfer equation for the photons

$$\frac{1}{c}\frac{\partial}{\partial t}I + \vec{n}.\nabla I = S_t(\nu, \vec{n}).$$

The right hand side depends on the opacities. We assume the opacities σ_a and σ_s are independent of the frequency

$$\sigma_a(\nu_0) = \sigma_a, \quad \sigma_s(\nu_0) = \sigma_s. \tag{1}$$

This hypothesis is called the grey hypothesis. Solving this grey equation on a Cartesian grid or even better on an unstructured grid is already a very difficult task. Therefore one relies on simplified equations. One makes the distinction between models and asymptotic limits. An asymptotic limit is obtained by using a particular scaling of the variables such that the equation can be rewritten as a simplified equation plus a small residual. Dropping the residual we get what we called an asymptotic limit. But it has been observed this is not enough. One needs to rely on more crude approximation to derive other sets of simplified equations. This method is not as rigorous as the first one. Essentially one drops some terms assuming, with physical intuition, that these terms are small. But it is not possible to prove that the partial differential equations can be approximated with a small residual. Such approximation is referred to as a model. In some sense an asymptotic limit is a model with a proof that the physically small terms are mathematically small.

Starting from a hierarchy of models (the moment model for radiation is one member of such a hierarchy), it is however possible to study the asymptotic limit of some models and to show that they have a limit and this limit is an asymptotic limit of the master transfer equation. We will show it is true for the moment model by comparison with the grey non equilibrium model which is an asymptotic limit. In this sense it justifies the model.

Section 3 is devoted to the presentation of basic numerical methods for the grey diffusion model and for the moment model in dimension one. This scheme is compatible with the Rankine-Hugoniot relations for the grey non equilibrium diffusion model.

In section 4 we discuss an advanced method based on the reformulation of moment model as compressible gas dynamics.

2. Hierarchy of grey models

2.1. Relativistic gas dynamics. – The Euler system of inviscid gas dynamics with full Lorentz invariance is [**26**]-[**30**]-[**35**]

$$(2)\qquad \begin{cases} \frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0, \\ \frac{\partial}{\partial t}(\gamma(1+\frac{h}{c^2})\rho\vec{v}) + \nabla.(\gamma(1+\frac{h}{c^2})\rho\vec{v}\otimes\vec{v} + p\mathbf{I}) = 0, \\ \frac{\partial}{\partial t}(\gamma\rho(1+\frac{h}{c^2}) - \frac{p}{c^2}) + \nabla.(\gamma\rho(1+\frac{h}{c^2})\vec{v}) = 0. \end{cases}$$

For this kind of Lorentz invariant models, there is a distinction between the Eulerian reference frame also referred to as the lab frame, and the comobile reference frame which moves with the fluid also referred to as the Lagrangian frame. The density $\rho = \frac{1}{\tau}$ in the lab frame is different from the density calculated in the comobile frame $\rho_0 = \frac{1}{\tau_0}$. In what follows the subscript 0 will designate any quantity measured in the comobile frame. One has $\rho = \gamma\rho_0$ where γ is defined by $\gamma = \frac{1}{\sqrt{1-\frac{|\vec{v}|^2}{c^2}}}$, c is the velocity of light. In (2) h is the enthalpy of the fluid calculated in the comobile frame $h = e_0 + p_0\tau_0$, where p_0 is the pressure. If one assumes for simplicity a perfect gas pressure law $p_0 = \Gamma\frac{e_0}{\tau_0}$, $\Gamma > 0$, then the enthalpy is simply $h = (\Gamma+1)e_0$. Here e_0 is the internal energy of the fluid calculated in the comobile frame. Multiplying the last equation of (2) with c^2 and subtracting the first one multiplied by c^2 for the sake of convenience, we rewrite it as

$$(3)\qquad \begin{cases} \frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0, \\ \frac{\partial}{\partial t}(\gamma(1+\frac{h}{c^2})\rho\vec{v}) + \nabla.(\gamma(1+\frac{h}{c^2})\rho\vec{v}\otimes\vec{v} + p_0\mathbf{I}) = 0, \\ \frac{\partial}{\partial t}(c^2\gamma\rho(1+\frac{e_0}{c^2}+\frac{|\vec{v}|^2}{c^2}\frac{p_0\tau_0}{c^2}) - c^2\rho) \\ \qquad +\nabla.(c^2(\gamma\rho(1+\frac{e_0}{c^2}+\frac{|\vec{v}|^2}{c^2}\frac{p_0\tau_0}{c^2}) - c^2\rho)\vec{v} + p_0\vec{v}) = 0. \end{cases}$$

For the sake of simplicity of notations, we define $\vec{v}_2 = \gamma(1+\frac{h}{c^2})\vec{v}$ and $E_2 = c^2\gamma(1+\frac{e_0}{c^2}+\frac{v^2}{c^2}\frac{p_0\tau_0}{c^2}) - c^2$. With these notations (3) is equivalent to

$$(4)\qquad \begin{cases} \frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0, \\ \frac{\partial}{\partial t}(\rho\vec{v}_2) + \nabla.(\rho\vec{v}_2\otimes\vec{v} + p_0\mathbf{I}) = 0, \\ \frac{\partial}{\partial t}(\rho E_2) + \nabla.(\rho E_2\vec{v} + p_0\vec{v}) = 0. \end{cases}$$

The structure of this Lorentz invariant system is close to the structure of the Galileo invariant system (6).

2.2. Galilean gas dynamics. – The classical Euler system of inviscid gas dynamics with Galilean invariance is recovered as the limit of (5) when $\frac{|\vec{v}|}{c} \to 0$. We consider the regime

$$\varepsilon = \frac{|\vec{v}|}{c}, \quad \frac{e_0}{c^2} = O(\varepsilon^2) \tag{5}$$

where ε is a small parameter. Indeed one has $\vec{v}_2 = \vec{v} + O(\varepsilon^2)$ and $E_2 = e_0 + \frac{1}{2}|\vec{v}|^2 + O(\varepsilon^2)$. Let us define the classical total energy $E = e_0 + \frac{1}{2}|\vec{v}|^2$. Then the classic Euler system of inviscid gas dynamics

$$\begin{cases} \frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0, \\ \frac{\partial}{\partial t}(\rho\vec{v}) + \nabla.(\rho\vec{v}\otimes\vec{v} + p\mathbf{I}) = 0, \\ \frac{\partial}{\partial t}(\rho E) + \nabla.(\rho E\vec{v} + p\vec{v}) = 0, \end{cases} \tag{6}$$

is recovered as the $O(\varepsilon^2)$ approximation of (2). In order to simplify the presentation and since we are interested mainly in flows moving at moderate velocities, we use (6) instead of (2) in the rest of this paper.

2.3. Grey transfer equation for photons. – The transfer equation for photons is

$$\frac{1}{c}\frac{\partial}{\partial t}I + \vec{n}.\nabla I = S_t(\nu, \vec{n}), \tag{7}$$

where $I(t, x, \nu, \vec{n})$ is the intensity of the radiation, ν the frequency and $\vec{n}$ the direction of the photons. The source term is $S_t(\nu, \vec{n})$. It is well known that the source term has to be Lorentz invariant for the total coupled system to be accurate [**30**]. We assume the grey hypothesis (1). This is a very crude approximation, but is motivated by the mathematical analysis. In this work we follow [**28**] and consider a simplified source term where $S_t = S_a + S_s$ is the sum of two contributions. The first one takes into account the absorption/re-emission of photons by the matter

$$S_a(\nu, \vec{n}) = \frac{\nu_0}{\nu}\sigma_a \left[\left(\frac{\nu}{\nu_0}\right)^3 B(\nu_0, T) - I \right]. \tag{8}$$

Here T is the material temperature, $B(\nu_0, T)$ is the Planckian

$$B(\nu_0, T) = \frac{2h\nu_0^3}{c^2}\left(e^{h\nu_0/kT} - 1\right)^{-1} \tag{9}$$

and $\sigma_a \geq 0$ is the absorption coefficient. In these definitions, one has to use the frequency and direction of the photon calculated in the comobile frame

$$\nu_0 = \gamma\nu(1 - \frac{\vec{n}.\vec{v}}{c}) \text{ and } \vec{n}_0 = \left(\frac{\nu}{\nu_0}\right)\left[\vec{n} - \frac{\gamma}{c}\vec{v}\left(1 - \frac{\vec{n}.\vec{v}}{c}\left(\frac{\gamma}{\gamma+1}\right)\right)\right]. \tag{10}$$

Another important invariance relation [**30**]-[**33**]-[**35**] between the intensity of the radiation in the lab frame and the intensity of the radiation in the comobile frame is

$$\frac{I}{\nu^3} = \frac{I_0}{\nu_0^3}. \tag{11}$$