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DISCRETE-ORDINATES TRANSPORT METHODS FOR NON-RELATIVISTIC RADIATION-HYDRODYNAMICS

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Abstract. – The discrete-ordinates method is a particular angular discretization of the radiative transfer equation that has become the dominant angular discretization over the last several decades. The term "discrete-ordinates methods" generally refers to the discrete-ordinates angular discretization applied in conjunction with various discretization techniques for the other variables in the transport equation, as well as solution techniques for the associated discrete equations. In this paper we present an overview of the application of discrete-ordinates methods to thermal radiation transport and radiation-hydrodynamics in the non-relativistic regime.

Résumé **(Méthodes des ordonnées discrètes pour l'hydrodynamique radiative non relativiste)**
La méthode des ordonnées discrètes consiste en une discrétisation angulaire particulière de l'équation du transfert radiatif. Cette méthode occupe depuis quelques années une position dominante. En fait, sous le vocable « méthodes des ordonnées discrètes » on fait référence à la discrétisation angulaire par ordonnées discrètes appliquée en combinaison avec des techniques de discrétisation variées pour les autres variables de l'équation de transport, ainsi qu'avec des techniques de résolution des équations discrètes associées. Dans cet article, nous présenterons une revue des applications de ces méthodes au transport de radiation et à l'hydrodynamique radiative en régime non-relativiste.

1. Introduction

The discrete-ordinates method is a particular angular discretization of the radiative transfer equation that has become the dominant angular discretization over the last several decades. The term "discrete-ordinates methods" generally refers to the discrete-ordinates angular discretization applied in conjunction with various discretization techniques for the other variables in the transport equation, as well as solution techniques for the associated discrete equations. The purpose of this paper is

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to briefly describe the application of discrete-ordinates methods to radiative transfer in the non-relativistic regime. We consider the non-relativistic regime to be characterized by material velocities less than or equal to one percent of the speed of light. In most applications, the radiative transfer equations are coupled to the hydrodynamics equations. We first describe a radiation-hydrodynamics model that consists of the Euler equations and the radiation transport equation together with approximations for material-motion effects based upon the assumption of non-relativistic material velocities. For simplicity, we next focus on a model of radiative transfer in a static medium that is adequate for the purpose of describing numerical discretization and solution techniques for the transfer equation that can also be applied in the more general context of radiation-hydrodynamics. Finally, we discuss operator splitting techniques intended to yield reliable radiation-hydrodynamics solutions using standard solution techniques for both the hydrodynamics equations and the radiative transfer equation.

2. An Approximate Radiation-Hydrodynamics Model

The hydrodynamics equations and the radiation moment equations can be expressed to $\mathrm{O}(v/c)$ as follows assuming a grey radiation transport approximation with the transfer equation cast in the Eulerian frame:

— *Conservation of fluid mass*:

$$\partial_t \rho + \partial_i \left(\rho v_i\right) = 0 \,, \tag{2.1}$$

— *Conservation of fluid momentum*:

$$\partial_t \left(\rho v_i\right) + \partial_j \left(\rho v_i v_j\right) + \partial_i p = \frac{\sigma_t}{c} F_{0,i} - \frac{v_i}{c}\sigma_a \left(aT^4 - E_0\right) \quad , \tag{2.2}$$

— *Conservation of total fluid energy*:

$$\begin{aligned} \partial_t \left(\frac{1}{2}\rho v^2 + \rho e\right) + \partial_i \left[\left(\frac{1}{2}\rho v^2 + \rho e + p\right) v_i\right] = \\ -c\sigma_a \left(aT^4 - E_0\right) + \frac{\sigma_t}{c} v_i F_{0,i} \quad , \end{aligned} \tag{2.3}$$

— *Conservation of radiation momentum*:

$$\frac{1}{c^2}\partial_t F_i + \partial_j P_{ij} = -\frac{\sigma_t}{c} F_{0,i} + \frac{v_i}{c}\sigma_a \left(aT^4 - E_0\right) \quad , \tag{2.4}$$

— *Conservation of radiation energy*:

$$\partial_t E + \partial_i F_i = c\sigma_a \left(aT^4 - E_0\right) - \frac{\sigma_t}{c} v_i F_{0,i} \quad , \tag{2.5}$$

where ρ is the fluid density, v_i is component i of the fluid velocity, c is the speed of light, p is the fluid pressure, e is the specific internal energy of the fluid, E is the radiation energy density, F_i is component i of the radiation flux, P_{ij} is element ij of the radiation pressure tensor, E_0 is the comoving-frame radiation energy density:

$$E_0 = E - \frac{2}{c^2} v_i \left(F_i - v_i E - v_j P_{ij}\right) \,, \tag{2.6}$$

$F_{0,i}$ is component i of the comoving-frame radiation flux:

$$F_{0,i} = F_i - v_i E - v_j P_{ij} \, , \tag{2.7}$$

σ_t is the macroscopic total cross section, σ_a is the macroscopic absorption cross section, and a is the radiation constant. The source terms in Eqs. (2.2) through (2.5) are expressed in terms of comoving-frame radiation variables. These comoving-frame variables are defined in terms of Eulerian-frame radiation variables by Eqs. (2.6) and (2.7). When the comoving-frame variables are eliminated from Eqs. (2.2) through (2.5) using Eqs. (2.6) and (2.7), certain terms of $\mathrm{O}(v^2/c^2)$ will appear. Even though Eqs. (2.2) through (2.5) are only correct to $\mathrm{O}(v/c)$, these terms are not to be neglected because they eliminate spurious equilibrium solutions [**15**] and maintain consistency between the Eulerian-frame and comoving-frame momentum and energy sources. We consider the nonrelativistic limit to be characterized by $v/c \leq 0.01$.

We next describe a very simple approximate model for radiation-hydrodynamics with multifrequency radiative transfer in the nonrelativistic limit. The goal of this approximation is to maintain accuracy in an integral sense while avoiding the complexities of the equations that are correct to $\mathrm{O}(v/c)$. Since v/c is very small in the nonrelativistic limit, one might be tempted to neglect material motion corrections to the radiative transfer equation entirely. However, over the course of a calculation, this can result in a significant loss of energy conservation because the kinetic energy change in the fluid due to radiation momentum deposition is not removed from the radiation energy field. Thus we use a simple approximation for nonrelativistic radiation-hydrodynamics that has the following properties:

— Total energy and momentum are conserved.
— Equilibrium solutions are preserved to $\mathrm{O}(\frac{v}{c})$.
— The equilibrium diffusion limit is preserved to $\mathrm{O}(\frac{v}{c})$.

In the nonrelativistic case, there is little difference between E and E_0, but the relative difference between $F_{0,i}$ and F_i can be arbitrarily large. This relative difference is essentially infinite in the equilibrium state because $F_{0,i} = 0$ while $F_i = \frac{4}{3} v_i E$ to $\mathrm{O}(v/c)$. Since the system is locally equilibrated in the equilibrium diffusion limit, it follows that one can expect a significant relative difference between $F_{0,i}$ and F_i in this limit. Hence it is generally considered important to preserve the equilibrium diffusion limit in any approximate treatment of nonrelativistic radiation-hydrodynamics. See [**15**] for a formal asymptotic derivation of the non-relativistic equilibrium diffusion limit.

The first step in the derivation of our approximate model is modify the hydrodynamics equations and the grey radiation moment equations. Specifically, we make the following substitutions:

$$\frac{\sigma_t}{c} F_{0,i} - \frac{v_i}{c} \sigma_a \left(aT^4 - E_0 \right) \Rightarrow \frac{\sigma_t}{c} F_{0,i} \ , \tag{2.8a}$$

$$c\sigma_a \left(aT^4 - E_0 \right) - \frac{\sigma_t}{c} v_i F_{0,i} \Rightarrow c\sigma_a \left(aT^4 - E \right) - \frac{\sigma_t}{c} v_i F_{0,i} \ , \tag{2.8b}$$

$$F_{0,i} = F_i - v_i E - v_j P_{ij} \Rightarrow F_i - \frac{4}{3} v_i E \ , \tag{2.8c}$$

The term neglected in (2.8a) is a purely relativistic term. Although the radiative transfer equation is always relativistic, the hydrodynamics equations are purely classical to $\mathrm{O}(v/c)$. Furthermore, this term is zero in the equilibrium diffusion limit. Thus it is reasonable to eliminate this term. In (2.8b) we replace E_0 with E in the radiation energy source term. It can be seen from Eq. (2.6) that this should be a good approximation when $v/c \leq 0.01$. Furthermore, $E_0 = E$ in the equilibrium diffusion limit. Finally, we substitute $(E/3)\,\delta_{i,j}$ for $P_{i,j}$ in (2.8c). This approximation is made simply to avoid calculating the radiation pressure tensor. This tensor plays a small role in a small term, and the substitution is exact in the equilibrium diffusion limit. The next step in the derivation is to replace the modified grey expressions with frequency-dependent expressions that are equivalent under the grey approximation. In particular,

$$\frac{\sigma_t}{c} F_{0,i} \Rightarrow \int_0^\infty \int_{4\pi} \frac{\sigma_t(\nu)}{c} I(\overrightarrow{\Omega}, \nu) \left(\Omega_i - \frac{4}{3} \frac{v_i}{c} \right) d\Omega \, d\nu \ , \tag{2.9a}$$

$$c\sigma_a \left(aT^4 - E \right) \Rightarrow \int_0^\infty \int_{4\pi} \sigma_a(\nu) \left[B(\nu) - I\left(\overrightarrow{\Omega}, \nu\right) \right] d\Omega \, d\nu \ , \tag{2.9b}$$

$$F_{0,i} \Rightarrow \int_0^\infty \int_{4\pi} I\left(\overrightarrow{\Omega}, \nu\right) \left(\Omega_i - \frac{4}{3} \frac{v_i}{c} \right) d\Omega \, d\nu \ , \tag{2.9c}$$

where $\overrightarrow{\Omega}$ is the photon direction vector, ν is the photon frequency, $I\left(\overrightarrow{\Omega}, \nu\right)$ is the angular intensity, $B = \frac{2h\nu^3}{c^2} \left[\exp\left(\frac{h\nu}{kT} \right) - 1 \right]^{-1}$ is the Planck function and k is the Boltzmann constant. Expression (2.9b) represents an exact substitution, and (2.9c) represents a substitution that is exact to $\mathrm{O}(v/c)$. Equation (2.9a) is formally incorrect unless σ_t is independent of frequency. The transformations between the Eulerian and comoving frames being used here are only correct for angle-energy-integrated quantities **[16]**. We nonetheless apply them to angle-energy-dependent quantities because this results in a significant simplification while retaining accuracy in an angle-energy-integrated sense. For instance, the transformation relating the comoving frame flux and the Eulerian frame flux is effectively being used for frequency-dependent fluxes in (Eq. (2.9a)).

Accounting for all of these substitutions, our approximate hydrodynamic equations and multifrequency radiation moment equations can be expressed as follows:

$$\partial_t \rho + \partial_i \left(\rho v_i \right) = 0 \ , \tag{2.10}$$