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Abstract. – The radiative transfer equations are considered in the present work to simulate the radiative fields and their interactions with the matter. The expected numerical simulations imply to couple radiative transfer and hydrodynamic. To perform such experiments, a radiative kinetic model is too expensive and moment models must be privileged. The M_1 model is thus considered and its main mathematical properties are established. A particular attention is paid on the asymptotic behavior of the model as soon as the opacities are large. Relevant numerical methods are then studied and proved to be robust and asymptotic preserving. Numerical experiments illustrate the interest of the M_1 model.

***Résumé* (Approximation numérique du modèle M_1).** – On s'intéresse ici à la simulation les interactions entre la matière et le transfert radiatif. Certaines applications nécessitent un couplage fort entre rayonnement et hydrodynamique complexe. Dans ce cas, il est très coûteux numériquement d'utiliser un modèle cinétique. Notre choix se porte alors sur un modèle aux moments: le modèle M_1. Ses principales propriétés mathématiques sont établies, en particulier le comportement asymptotique du modèle lorsque les opacités deviennent grandes. Des méthodes numériques sont ensuite étudiées et leur robustesse ainsi que leur comportement asymptotique sont alors établis. Enfin, on illustre l'intérêt du modèle M_1 par des simulations numériques.

1. Introduction

Radiative transfer may have a huge impact on the hydrodynamic flow in applications such as superorbital atmospheric re-entry, fires or astrophysics. In such regimes, full coupling between material and radiation is mandatory. However, the use of a model that solves the full Radiative Transfer Equation (RTE) for these kind of fully coupled multidimensional simulations also involving other complex phenomena such as chemical reactions or turbulence is way beyond computer limits. Therefore, it is

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important to develop cheaper radiative models that are accurate, robust and preserve the important physical behaviors.

In this context, the M_1 model is an interesting choice. It is a moments model ie integrated over all directions and frequencies (or frequency intervals for multigroup versions) and hence has a relatively low computational cost. It also has several very interesting properties as we will see below. The M_1 model was first introduced in **[10]** and the multigroup version in **[25]**.

In this paper, we introduce the M_1 model and derive several appropriate solvers. The robustness of the numerical approximation will be one of the main focus as well as the conservation of the important physical properties. Particular attention will be paid to preserve the energy positiveness, the flux limitation, the total energy conservation. Moreover, we carefully check that the scheme is asymptotic preserving in order to ensure the relevance of the numerical approximations.

To access such an issue, finite volume techniques are considered. In the framework of the M_1 model for the radiative transfer, recent numerical developments based on HLL's **[15]** scheme and Suliciu's relaxation scheme **[5]** are presented.

The paper is organized as follows: the M_1 model is introduced in the next section along with its main properties. Three schemes are developed and analyzed in sections 3 to 5. To conclude, two numerical experiments are given to illustrate the interest of the M_1 model and its associated numerical schemes.

2. The M_1 model

This section is dedicated to the construction of the M_1 model and its most important properties.

We start by considering the time-dependent Radiative Transfer Equation in local thermal equilibrium:

$$(1)\quad \frac{1}{c}\partial_t I_\nu(\Omega)+\Omega.\nabla_x I_\nu(\Omega) = \sigma_\nu B_\nu(T)-(\sigma_\nu^d+\sigma_\nu)I_\nu(\Omega)+\frac{\sigma_\nu^d}{4\pi}\int_{S^2} p_\nu(\Omega'.\Omega)I_\nu(\Omega')d\Omega' d\nu,$$

where c is the speed of light, T the material temperature, Ω the photons direction of propagation, ν the frequency and σ^d, σ are respectively the scattering and absorption opacities. Moreover, $B_\nu(T)$ is Plank's blackbody function $B_\nu(T)$ as:

$$(2)\quad B_\nu(T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}.$$

The radiative intensity $I_\nu(\Omega) = I(t,x,\Omega,\nu)$ is linked with the photons' distribution function. It is a function of seven variables in 3D **[18]**, **[20]**.

Finally, $p_\nu/4\pi$ is the scattering probability density. Several choices of probabilities may be considered depending on the application. For instance, a classical choice in plasmas physics is expressed by the Thomson kernel:

$$p_\nu(\Omega.\Omega') = p(\Omega.\Omega') = C(1+(\Omega.\Omega')^2),\ C\in\mathbb{R}.$$

Moments models are constructed through integrations of the RTE over frequencies and directions. While these integrations are usually performed together, we will use two steps in this paper to show the respective role of integrations over directions and frequencies.

The first step is dedicated to the construction of an intermediate model integrated over directions from which we perform the second step. The full M$_1$ model and its multigroup version are the result of this second step.

First, integrations of (1) over all the directions lead to the following intermediate model:

$$\partial_t E(\nu) + \nabla_x . F(\nu) = c\sigma_\nu \Big(4\pi B_\nu(T) - E(\nu)\Big) \tag{3}$$

$$\partial_t F(\nu) + c^2 \nabla_x . P(\nu) = -c(\sigma_\nu + \sigma_\nu^d (1 - \check{g}_\nu))F(\nu) \tag{4}$$

The first three angular moments of the radiative intensity I are $E(\nu)$, $F(\nu)$ and $P(\nu)$. Moreover, $\check{g}_\nu$ is the first moment of the scattering operator (for Thomson scattering $\check{g}_\nu = 0$).

Since there are more unknowns than equations, one needs an additional hypothesis to solve this system. Indeed $P(\nu)$ is taken as a function of $E(\nu)$ and $F(\nu)$ and the choice of this closure function determines the model. The classical $P1$ choice of closure is to set $P(\nu) = E(\nu)/3$. This choice is perfectly valid close to the radiative equilibrium but leads to unphysical results far from it. Various other closure functions were introduced in the literature, e.g. [**19**].

The M$_1$ closure function is given by a minimum entropy principle. This technique was introduced by D. Levermore for fluid mechanics [**17**]. It leads to the following closure function: $\mathscr{I}(\Omega, \nu) = \dfrac{2h\nu^3}{c^2}[\exp(\dfrac{h\nu}{k}\boldsymbol{m}.\boldsymbol{\alpha}(\nu)) - 1]^{-1}$ where $\boldsymbol{\alpha}(\nu)$ is a Lagrange multiplier and $\boldsymbol{m} = (1, \Omega)^\top$. This closure function $\mathscr{I}$ is the underlying radiative intensity predicted by the model instead of the real solution I of the RTE. The system is closed by enforcing the radiative pressure $P(\nu)$ to be the second moment of the closure function $\mathscr{I}$:

$$P(\nu) = \frac{1}{c}\int_{S^2} \Omega \otimes \Omega \mathscr{I}(\Omega, \nu) d\Omega.$$

It is possible to write the resulting radiative pressure in Eddington form: $P(\nu) = D(\nu)E(\nu)$. Eddington's tensor $D(\nu)$ is then given by:

$$D(\nu) = \frac{1 - \chi(\nu)}{2} I_d + \frac{3\chi(\nu) - 1}{2} \check{f}(\nu) \otimes \check{f}(\nu) \tag{5}$$

where: $\check{f}(\nu) = \dfrac{F(\nu)}{\|F(\nu)\|}$ and $\chi(\nu)$ is the eigenvalue of $D(\nu)$ associated with $\check{f}(\nu)$.

The intermediate model (3)-(4) closed thanks to the minimum entropy principle and coupled to a relevant material temperature equation is a symetrizable hyperbolic system which locally decreases the total entropy and conserves the energy. Moreover, $\boldsymbol{\alpha}(\nu)$ exists and is unique as soon as $(E(\nu), F(\nu))$ is physically relevant.

Of course, for computing purposes, we have to go one step further in the modeling. Considering a finite number of frequency groups and integrating inside each group the intermediate model, we get the following system:

$$\partial_t E_q + \nabla_x . F_q = c[\sigma_q^e a \theta_q^4(T) - \sigma_q^a E_q], \ 1 \leq q \leq Q \tag{6}$$

$$\frac{1}{c}\partial_t F_q + c\nabla_x . P_q = (-\sigma_q^f - \sigma_q^d(1 - \breve{g}_q))F_q, \ 1 \leq q \leq Q \tag{7}$$

Definition 1. – *For the sake of simplicity, the following notation will be used:*

$$<.>_q = \frac{1}{c}\int_{S^2}\int_{\nu_{q-\frac{1}{2}}}^{\nu_{q+\frac{1}{2}}} .d\nu d\Omega$$

With the previous definition, we have $\theta_q(T) = \frac{1}{a}<B>_q^{1/4}$.

Usually, a group is a "large" frequency interval. A typical multigroup computation would consider up to a few dozen groups.

The first three moments of the radiative intensity I are:

$$E_q = <I>_q \tag{8}$$

$$F_q = <c\Omega I>_q \tag{9}$$

$$P_q = <\Omega \otimes \Omega I>_q. \tag{10}$$

In the system (6)-(7), σ_q^a, σ_q^e, σ_q^f and σ_q^d are the opacities' mean values inside the q^{th} group. To complete the definition of (6)-(7) we have:

$$\sigma_q^e a \theta_q^4(T) = <\sigma B(T)>_q,$$
$$\sigma_q^a E_q = <\sigma I>_q,$$
$$\sigma_q^f F_q = <c\sigma\Omega I>_q.$$

The computation of these mean values may be very difficult for realistic computations and is critical for the precision of the solutions. We will see later (see section 2.3) how they can be handled by the M_1 model but in this section only constant per group opacities are considered for the sake of simplicity.

Definition 2. – *The radiative entropy density of the considered system is:*

$$\breve{h}(I) = \frac{2k\nu^2}{c^3}\Big[n_I \ln n_I - (n_I + 1)\ln(n_I + 1)\Big], \tag{11}$$

where n_I is the occupation number given by: $n_I = \frac{c^2}{2h\nu^3} I$ and the entropy is given by:

$$H(I) = <\breve{h}(I)> = \sum_q <\breve{h}(I)>_q. \tag{12}$$

It is to note that Planck's function is the minimum of the radiative entropy:

$$B_\nu(T) = \operatorname{argmin}\{H(I), <I> = aT^4\}.$$