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SHOCK PROFILES IN RADIATIVE HYDRODYNAMICS

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Abstract. – In this article we review some models and recent results in radiative hydrodynamics. We first focus on the derivation of asymptotic models. Then we present some results on the analysis of a scalar toy model. These first results guide us through the analysis of a more realistic model that couples the hydrodynamics equations with a diffusion equation for the radiations. In the end, we present some numerical procedures that capture traveling waves solutions.

***Résumé* (Profils de choc en hydrodynamique radiative).** – Dans cette contribution nous présentons quelques résultats récents sur des modèles d'hydrodynamique radiative. Tout d'abord, nous nous intéressons à la dérivation de modèles asymptotiques. Puis nous discutons quelques résultats sur un modèle simplifié scalaire. Ces résultats servent de guide pour l'analyse d'un modèle plus réaliste qui couple un système d'équations hydrodynamiques avec une équation de diffusion pour les radiations. Nous terminons en détaillant des procédures de simulation numérique qui permettent de capturer les solutions de type ondes progressives.

1. Introduction

There exists a huge variety of models intended to describe, with more or less accuracy, radiative transfer phenomena. Of course these models are more or less complex, and retain more or less details of the physics. The choice of a model among the list is definitely application-dependent, driven by the leading phenomena in the considered situation, thus depending on the values of the physical parameters arising in the equations. However, it is not completely clear how to draw a complete hierarchy, based on asymptotics arguments. A basis model couples the evolution of quantities describing the fluid and the radiation which interact through energy (and also possibly momentum) exchanges. To start with, we can describe the fluid by its density $\rho(t, x) \geq 0$,

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its velocity field $u(t,x) \in \mathbb{R}^3$ and its total energy $E(t,x) \geq 0$, which are functions of the time variable $t \geq 0$ and position variable $x \in \mathbb{R}^3$. Assuming an equation of state for the pressure, these quantities should satisfy the relativistic Euler system. Next, radiations are described by their specific intensity $f(t,x,v,\nu)$ that also depends on the frequency $\nu > 0$ and on the variable $v \in \mathbb{S}^2$ that represents the direction of flight of the photons. It has the dimension of an energy per (surface$\times$time$\times$ frequency) units so that:

$$\frac{1}{c}\int_{\mathscr{V}\times\mathscr{N}\times\mathscr{O}} f(t,x,v,\nu)\,\mathrm{d}v\,\mathrm{d}\nu\,\mathrm{d}x$$

where c is the speed of light, gives the radiant energy at time $t \geq 0$ corresponding to photons emitted from a position $x \in \mathscr{O} \subset \Omega$, flying in direction $v \in \mathscr{V} \subset \mathbb{S}^2$, and having a frequency $\nu \in \mathscr{N} \subset (0,\infty)$. We can equivalently work with the distribution function $F(t,x,p)$ which gives the number of photons per unit volume in phase space. These quantities are related by the simple change of variables formulae:

$$p = \frac{h\,\nu}{c}\,v, \qquad h\,\nu\,F\,\mathrm{d}p\,\mathrm{d}x = \frac{1}{c}\,f\,\mathrm{d}\nu\,\mathrm{d}v\,\mathrm{d}x\,,$$

where h stands for the Planck constant. Since $\mathrm{d}p = \left(\frac{h}{c}\right)^3 \nu^2\,\mathrm{d}\nu\,\mathrm{d}v$, we finally get:

$$f(t,x,v,\nu) = \frac{h^4\,\nu^3}{c^2}\,F\big(t,x,\frac{h\nu}{c}v\big)\,.$$

The evolution of the specific intensity of radiation is governed by a kinetic equation:

$$\frac{1}{c}\partial_t f + v\cdot\nabla_x f = Q\,. \tag{1}$$

The left hand side describes the fact that photons travel on straight lines at the speed of light but this simple motion is perturbed by interaction processes embodied into the right hand side of (1). The leading phenomena are the following:

- Scattering produces a change in the direction of flight, due for instance to collisions with atoms of the fluid.

- Emission and absorption correspond to energy exchanges between the fluid and the radiation fields.

Accordingly, we split the interaction operator as follows: $Q = Q_{\mathrm{em/abs}} + Q_{\mathrm{scat}}$. Starting from such a complicated modeling, various simplifications are commonly used in practice:

- We can neglect the dependence of the coefficients of the interaction operator with respect to the frequency variable. In turn, we can average the equation with respect to ν. This is the so-called "grey assumption". From now on, we adopt this restriction, still referring to (1) but with an unknown f that depends on (t,x,v) only.
- Relativistic effects in the Euler equations are neglected, but in the coupling terms, since they only produce terms of order $\mathscr{O}((|u|/c)^2)$.

– We consider hydrodynamic regimes for describing the radiation, which allows to reduce again the size of the variables of the problem: in such regimes, the dependence with respect to the variable v is imposed and radiations are only described by the evolution of a macroscopic quantity that will be denoted $n(t,x) = \int_{\mathbb{S}^2} f \,\mathrm{d}v$.

Actually, we distinguish at least two relevant asymptotic regimes:

– In the equilibrium regime emission-absorption dominates so that the temperature of the radiation (that is typically $n^{1/4}$) relaxes to the temperature θ of the fluid. Hence, we are led to a system of PDEs whose unknowns are ρ, u, θ.
– In the non equilibrium regime scattering is the leading phenomena so that the intensity of radiation relaxes to a equilibrium state which defines a temperature that is different from the fluid temperature. Accordingly, we are led to a system of PDEs whose unknowns are ρ, u, θ (or E) and n.

Furthermore, the asymptotic regimes can retain some influence of the relativistic effects. Indeed, in the interaction term, some corrections of order $\mathscr{O}(|u|/c)$ appear. These corrections are due to the differences in the measurements made in the reference frame or by an observer moving with a certain velocity with respect to that lab frame. In particular, the scattering is conservative (or isotropic) in the co-moving frame, but not in the lab frame: $\int_{\mathbb{S}^2} Q_{\mathrm{scat}} \,\mathrm{d}v \neq 0$. It turns out that these Doppler corrections may imply convection terms in the diffusion regimes, depending on the scaling.

Concerning introduction to the physics, and details on the coupling with hydrodynamics we refer to the classical treatise by Mihalas-Mihalas [**26**]. A recent detailed description of the scaling issues can be found in Lowrie-Hittinger-Morel [**25**] and Buet-Després [**4**] who bring out the effects of Doppler corrections. The mathematical analysis of the coupled problem Euler-Kinetic would be certainly very delicate and it seems difficult to expect more than local in time existence of smooth solutions; such an analysis, based on the classical fixed point approach for hyperbolic systems, has been performed in e.g. [**31, 22**]. For a justification of the diffusion asymptotics on simplified models, we refer to [**2, 12, 1**] and the references therein. Let us also mention [**9**] that deals with both equilibrium and non-equilibrium regimes, a topic that is also investigated in [**11**], adding to the model the influence of Doppler corrections.

In what follows, as in the other contributions to the present volume, we focus on the so-called "non equilibrium regime". The system of PDEs can be written in the following dimensionless form:

$$(2)\qquad \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x(\rho u \otimes u) + \partial_x p = -\mathscr{P}\Big(\dfrac{1}{\mathscr{L}_s}\langle v\, Q_{\mathrm{scat}}\rangle + \dfrac{1}{\mathscr{L}_a}\langle v\, Q_{\mathrm{em/abs}}\rangle\Big), \\ \partial_t(\rho E) + \nabla_x \cdot (\rho E u + p u) = -\mathscr{P}\mathscr{C}\Big(\dfrac{1}{\mathscr{L}_s}\langle Q_{\mathrm{scat}}\rangle + \dfrac{1}{\mathscr{L}_a}\langle Q_{\mathrm{em/abs}}\rangle\Big), \\ \dfrac{1}{\mathscr{C}}\,\partial_t f + v \cdot \partial_x f = \dfrac{1}{\mathscr{L}_s}\, Q_{\mathrm{scat}} + \dfrac{1}{\mathscr{L}_a}\, Q_{\mathrm{em/abs}}, \end{cases}$$

where we denote:

$$\langle f \rangle = \int_{\mathbb{S}^2} f \, \mathrm{d}v \,,$$

the pressure p and the total energy E are given by the perfect gas constitutive laws:

$$p = R\rho\theta \,, \quad E = \frac{1}{2}\,|u|^2 + \frac{R}{\gamma - 1}\,\theta \,,$$

with R, γ some positive constants, and θ the temperature. Eventually, the dimensionless parameters in (2) are:

- $\mathscr{P}$, which compares the typical energy of the radiations and the typical energy of the gas, is kept constant,
- $\mathscr{C}$, the ratio of the speed of light over the typical sound speed of the gas; $\mathscr{C}$ is typically large and will be denoted $1/\varepsilon \gg 1$ where ε is a small scaling parameter,
- $\mathscr{L}_{a,s}$, the Knudsen numbers associated to absorption/emission and scattering respectively; here we consider situations where scattering phenomena dominate over emission/absorption meaning that scattering phenomena take place on a much faster time scale that emission/absorption; we thus consider parameters $\mathscr{L}_a = 1/\varepsilon \gg 1$, and $\mathscr{L}_s = \varepsilon \ll 1$.

This "non equilibrium diffusion scaling" is relevant for instance in astrophysics when dealing with very rarefied and hot atmospheres, see **[30, 26]**. Intuitively, as $\varepsilon \to 0$, f_ε tends to make the scattering operator vanish. To make the discussion clear, let us write a possible expression for the scattering and emission/absorption operators:

$$Q_{\mathrm{scat}} = \sigma_s \left(\frac{1}{\Lambda_\varepsilon(v)^3} \int_{\mathbb{S}^2} \Lambda_\varepsilon(v')^2 f(v') \, \mathrm{d}v' - \Lambda_\varepsilon(v) f(v) \right),$$
$$Q_{\mathrm{em/abs}} = \sigma_a \left(\frac{\theta^4}{\Lambda_\varepsilon^3} - \Lambda_\varepsilon f \right),$$

with given constant coefficients $\sigma_s > 0$, and $\sigma_a > 0$. Of course, more complicated and non-isotropic operators are possible, but this model is enough for our purposes. The Doppler corrections are accounted for through the coefficient

$$\Lambda_\varepsilon(t,x,v) = \frac{1 - \varepsilon u(t,x) \cdot v}{\sqrt{1 - \varepsilon^2 |u(t,x)|^2}},$$

which involves the velocity $u(t,x)$ of the fluid. Neglecting relativistic effects this coefficient is replaced by 1. As a matter of fact, we observe that $\langle \Lambda_\varepsilon \, Q_{\mathrm{scat}} \rangle = 0$ (scattering is conservative in the co-mobile frame) and $\mathrm{Ker}(Q_{\mathrm{scat}}) = \mathrm{Span}(\Lambda_\varepsilon^{-4})$. In particular, the flux associated to this equilibrium distribution does not vanish: $\int_{\mathbb{S}^2} v \, \Lambda_\varepsilon^{-4} \, \mathrm{d}v \neq 0$, and this $\mathscr{O}(\varepsilon)$ quantity produces some unusual additional terms in the limit equations. Therefore, as ε goes to 0, f_ε becomes proportional to $\Lambda_\varepsilon^{-4} = \big(\frac{1 - \varepsilon u(t,x) \cdot v}{\sqrt{1 - \varepsilon^2 u^2}}\big)^{-4}$, up to small remainder terms. We can guess the limit equation by using the following Hilbert expansion:

$$f_\varepsilon = f^{(0)} + \varepsilon\, f^{(1)} + \varepsilon^2\, f^{(2)} + \dots \tag{3}$$