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WEAK APPROXIMATION AND RATIONALLY CONNECTED VARIETIES OVER FUNCTION FIELDS OF CURVES

by

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Abstract. – This survey addresses weak approximation for rationally connected varieties over function fields of complex curves. Topics include weak approximation at places of good reduction and the impact of mildly singular fibers; results for surfaces, varieties of low degree, and Fano hypersurfaces with square-free discriminant; and implications of rational simple connectedness for weak approximation. An appendix summarizes basic properties of moduli spaces of stable maps that are useful in the study of rationally connected varieties.

***Résumé* (Approximation faible et variétés rationnellement connexes sur des corps de fonctions de courbes)**

Cet article de synthèse porte sur l'approximation faible pour les variétés rationnellement connexes sur les corps de fonctions de courbes complexes. On y explique comment l'approximation faible vaut aux places de bonne réduction et ce qui se passe aux fibres sans singularités excessives. On discute le cas des surfaces, des variétés de petit degré, et des hypersurfaces de Fano de discriminant sans facteur carré. On explique aussi comment la simple connexité rationnelle donne des résultats sur l'approximation faible. En appendice on résume les propriétés des espaces de modules d'applications stables qui sont utilisées dans l'étude des variétés rationnellement connexes.

Introduction

This paper surveys recent work on weak approximation for varieties over complex function fields. It also touches on the geometric theory of rationally connected varieties and stable maps.

Weak approximation has been studied extensively in the context of number theory, quadratic forms, and linear algebraic groups. Early examples include work of Kneser

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[27] and Harder [15] on linear algebraic groups over various fields. In the 1980's, attention shifted to rational surfaces over number fields and cohomological obstructions to weak approximation. Significant results were obtained by Colliot-Thélène, Sansuc, Swinnerton-Dyer, Skorobogatov, Salberger, Harari, and others. We refer the reader to [14] for an excellent survey of the state of this area in 2002.

With the development of the theory of rationally connected varieties, weak approximation over function fields of *complex* curves became a focus of research. Already in 1992, Kollár-Miyaoka-Mori [31] showed that rationally connected varieties over such fields enjoy remarkable approximation properties, assuming they admit rational points. In 2001, Graber-Harris-Starr [13] showed these rational points exist, which opened the door to a more systematic study of their properties.

This paper is organized as follows: Section 1 reviews the basic definitions, presenting them in a form useful for our purposes. In Section 2, we present results valid for general rationally connected varieties, as well as key constructions and deformation-theoretic tools. We turn to special classes of varieties in Section 3, including rational surfaces and hypersurfaces with mild singularities at places of bad reduction. Section 4 addresses a large class of varieties where weak approximation is known, the rationally simply connected varieties [24]. We raise some questions for further study in Section 5. The Appendix presents basic facts on stable maps used throughout the volume.

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My personal research contributions in this area are in collaboration with Yuri Tschinkel, who made helpful suggestions on the content and presentation. Jason Starr made very important contributions to the arguments presented in Section 4 linking weak approximation to rational simple connectedness. I benefited from comments from the other speakers at the school, Laurent Bonavero and Olivier Wittenberg, and from the participants, especially Amanda Knecht and Chenyang Xu. I am grateful to Colliot-Thélène for his constructive comments on drafts of this paper.

1. Elements of weak approximation

Notation. – Throughout, a *variety* over a field L designates a separated geometrically integral scheme of finite type over L; its *generic point* is the unique point corresponding to its function field. A *general point* of a variety is a closed point chosen from the complement of an unspecified Zariski closed proper subset.

Let k be an algebraically closed field of characteristic zero and B a smooth projective curve over k, with function field $F = k(B)$.

Let X be a smooth projective variety over F. A *model* of X is a flat proper morphism $\pi : \mathcal{X} \to B$ with generic fiber X. Usually, $\mathcal{X}$ is a scheme projective over B, but there are situations where we should take it to be an algebraic space proper over B. For each $b \in B$, let $\mathcal{X}_b = \pi^{-1}(b)$ denote the fiber over b. Once we have chosen a concrete embedding $X \subset \mathbb{P}^N$, the properness of the Hilbert scheme yields a natural model. The model is *regular* if the total space $\mathcal{X}$ is nonsingular; this can always be achieved via resolution of singularities.

Elementary properties of sections. – Recall that a *section* of π is a morphism $s : B \to \mathcal{X}$ such that $\pi \circ s : B \to B$ is the identity. By the valuative criterion of properness, we have

$$\{\text{ sections } s : B \to \mathcal{X} \text{ of } \pi\} \Leftrightarrow \{\text{ rational points } x \in X(F)\}.$$

Assume $\mathcal{X}$ is regular and write

$$\begin{aligned} \mathcal{X}^{sm} &= \{x \in \mathcal{X} : \pi \text{ is smooth at } x\} \\ &= \{x \in \mathcal{X} : \mathcal{X}_b \text{ is smooth at } x, b = \pi(x)\} \subset \mathcal{X}. \end{aligned} \tag{1}$$

Then each section $s : B \to \mathcal{X}$ is necessarily contained in $\mathcal{X}^{sm}$. The proof of this assertion is basic calculus: Since $\pi \circ s$ is the identity the derivative $d(\pi \circ s)$ is as well, which means that $d\pi$ is surjective and

$$\dim \ker(d\pi_{s(b)}) = \dim \mathcal{X}_b$$

for each $b \in B$. Thus $\mathcal{X}_b$ is smooth at $s(b)$.

Formulating weak approximation. – For each $b \in B$, let $\widehat{\mathcal{O}}_{B,b}$ denote the completion of the local ring $\mathcal{O}_{B,b}$ at the maximal ideal $\mathfrak{m}_{B,b}$, and $\widehat{F}_b$ the completion of $F = k(B)$ at b, i.e., the quotient field of $\widehat{\mathcal{O}}_{B,b}$. Consider the *adèles* over F

$$\mathbb{A}_F = \prod_{b \in B}{}' \widehat{F}_b,$$

i.e., the restricted product over all the places of B. The restricted product means that all but finitely many of the factors are in $\widehat{\mathcal{O}}_{B,b}$. There are *two* natural topologies one could consider: The ordinary product topology and the restricted product topology, with basis consisting of products of open sets $\prod_{b \in B} U_b$, where $U_b = \widehat{\mathcal{O}}_{B,b}$ for all but finitely many b. Using the natural inclusions $F \subset \widehat{F}_b$, we may regard $F \subset \mathbb{A}_F$.

Given a variety X over F, the adelic points $X(\mathbb{A}_F)$ inherit both topologies from $\mathbb{A}_F$.

Definition 1.1. – A variety X over F satisfies *weak approximation* (resp. *strong approximation*) if

$$X(F) \subset X(\mathbb{A}_F)$$

is dense in the ordinary product (resp. restricted product) topology.

For *proper* varieties X, the distinction between weak and strong approximation is irrelevant. This will be clear after we analyze the definitions in this case.

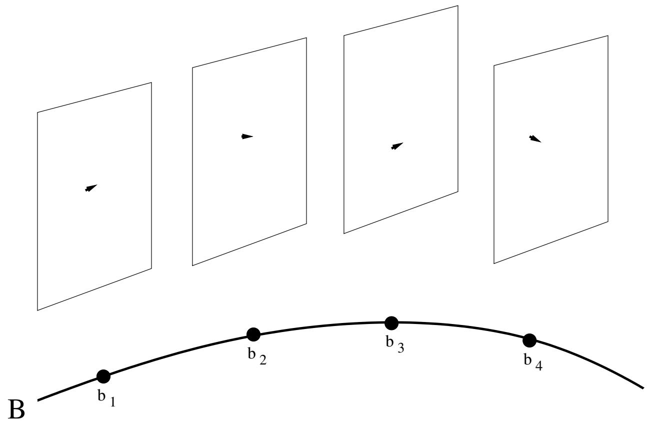

FIGURE 1. Jet data for sections

Unwinding the definition. – Assume X is smooth and proper and fix a model $\pi : \mathcal{X} \to B$. Both topologies on $X(\mathbb{A}_F)$ have the following basis: Consider data

$$J = (N; b_1, \ldots, b_r; \hat{s}_1, \ldots, \hat{s}_r),$$

consisting of a nonnegative integer N, distinct places $b_1, \ldots, b_r \in B$, and points $\hat{s}_i \in X(\widehat{F}_{b_i})$ for $i = 1, \ldots, r$. Since $\pi : \mathcal{X} \to B$ is proper, we may interpret $\hat{s}_i$ as a section of the restriction

$$\pi|\widehat{B}_{b_i} : \mathcal{X} \times_B \widehat{B}_{b_i} \to \widehat{B}_{b_i}, \quad \widehat{B}_{b_i} = \operatorname{Spec}(\widehat{\mathcal{O}}_{B,b_i}),$$

to the completion of B at b_i. Since we can freely clear denominators, insisting that the points are integral at almost all places is not a restriction. Thus our basic open sets are

$$U_J = \{t \in X(\mathbb{A}_F) : t \equiv \hat{s}_i \pmod{\mathfrak{m}_{B,b_i}^{N+1}}\},$$

i.e., sections with Taylor series at $b_1, \ldots, b_r$ prescribed to order N.

Now suppose in addition that $\pi : \mathcal{X} \to B$ is a *regular* model, so that sections automatically factor through $\mathcal{X}^{sm} \subset \mathcal{X}$ (see (1) above). Note that $\hat{s}_i(b_i) \in \mathcal{X}_{b_i}$ is a smooth point, by the same calculus argument we used to show sections factor through $\mathcal{X}^{sm}$. Conversely, Hensel's Lemma (or the $\mathfrak{m}$-adic version of Newton's method, cf. [**38**, p.14]) implies that each section $\hat{s}_i^N$ of

$$\mathcal{X}^{sm} \times_B \operatorname{Spec}(\mathcal{O}_{B,b_i}/\mathfrak{m}_{B,b_i}^{N+1}) \to \operatorname{Spec}(\mathcal{O}_{B,b_i}/\mathfrak{m}_{B,b_i}^{N+1})$$

can be extended to a section $\hat{s}_i$ of $\pi|\widehat{B}_{b_i}$. Thus we can recast our data as a collection of *jet data*

$$J = (N; b_1, \ldots, b_r; \hat{s}_1^N, \ldots, \hat{s}_r^N), \tag{2}$$

where the $\hat{s}_i^N$ are N-jets of sections of $\mathcal{X}^{sm} \to B$ at b_i.

To summarize: