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# RATIONAL POINTS OF RATIONALLY SIMPLY CONNECTED VARIETIES

*by*

Jason Michael Starr

***Abstract.*** – This is a survey of joint work with A. J. de Jong and Xuhua He proving Serre's "Conjecture II" for a function field $K$ of a surface over an algebraically closed field $k$ in the *split case*: for every simply connected, semisimple algebraic group $G$ over $k$, every $G$-torsor over $K$ has a $K$-point. This follows from a theorem, in characteristic 0, saying that a smooth, projective variety $X$ over $K$ has a $K$-point if the *elementary obstruction* of Colliot-Thélène and Sansuc vanishes, and if certain additional hypotheses hold—most importantly the geometric generic fiber must be rationally simply connected and must have a very twisting surface.

***Résumé* (Points rationnels des variétés rationnellement simplement connexes).** – Nous présentons un rapport détaillé sur un travail commun avec A. J. de Jong et Xuhua He, travail qui établit, dans le cas *déployé*, la « conjecture II » de Serre sur un corps $K$ de fonctions de deux variables sur un corps $k$ algébriquement clos : pour tout groupe algébrique $G$ semisimple simplement connexe sur $k$, tout $G$-torseur sur $K$ a un $K$-point. C'est une conséquence d'un théorème (en caractéristique nulle) selon lequel une variété $X$ projective et lisse sur $K$ possède un $K$-point si d'une part il n'y a pas d'*obstruction élémentaire* (comme définie par Colliot-Thélène et Sansuc) et si d'autre part certaines hypothèses géométriques sont satisfaites – les plus importantes étant que la fibre générique géométrique est rationnellement simplement connexe, et qu'elle possède une surface très tordante.

## 1. Introduction

The goal of these notes is to present some new results proved jointly with A. J. de Jong and Xuhua He. First, an algebraic fibration over a surface has a rational section if the fiber is "rationally simply connected" and if the *elementary obstruction* of Colliot-Thélène and Sansuc vanishes. Second, this implies the split, geometric case of a conjecture of Serre, "Conjecture II" in [**27**, p. 137].

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***Theorem 1.1***. – *For a connected, simply connected, semisimple algebraic group $G$ over an algebraically closed field $k$, every principal $G$-bundle over a $k$-surface has a rational section.*

Many others have worked towards the resolution of Serre's "Conjecture II" in the geometric case and in the general case: Merkurjev and Suslin; E. Bayer and R. Parimala; Chernousov; and P. Gille. These results are summarized in [**5**, Theorem 1.2(v)]. Because of these many results, the full "Conjecture II" in the geometric case reduces to the split, geometric case, so that "Conjecture II" is now settled in the geometric case.

These notes closely follow our article [**20**]. But the arguments here are a bit simpler, and the hypotheses are considerably stronger (yet still verified in the application to Serre's conjecture).

These notes accompany lectures delivered at the conference *Variétés rationnellement connexes: aspects géométriques et arithmétiques* of the Société Mathématique de France held in Strasbourg, France in May 2008. In addition to the new results, the lectures also presented the proof of the Kollár-Miyaoka-Mori conjecture proved by Tom Graber, Joe Harris and the author in characteristic 0 and by A. J. de Jong and the author in arbitrary characteristic. But as there are already several expositions of that work, I will only review the main statement.

***Theorem 1.2***. – [**11**], [**18**] *Let $k$ be an algebraically closed field. Let $C$ be a smooth, projective $k$-curve. Let $\pi : X \to C$ be a projective, flat morphism. Assume that the geometric generic fiber $Y$ of $\pi$ is irreducible and normal and that the smooth locus of $Y$ is* separably rationally connected, *i.e., there exists a morphism from $\mathbb{P}^1$ into the smooth locus such that the pullback of $T_Y$ is ample on $\mathbb{P}^1$. Then there exists a $k$-morphism $\sigma : C \to X$ such that $\pi \circ \sigma = \mathrm{Id}_C$, i.e., $\sigma$ is a section of $\pi$.*

Here is the formulation of the split, geometric case of Serre's Conjecture II which we prove here.

***Corollary 1.3***. – *Let $\kappa$ be an algebraically closed field of characteristic 0. Let $S$ be a smooth, integral, projective surface over $\kappa$. Let $f : X \to S$ be a proper, surjective morphism. Assume there exists a Zariski open subset $U$ of $S$ and an invertible sheaf $\mathscr{L}$ on $f^{-1}(U)$ such that*

(i) *$S - U$ is a finite collection of $\kappa$-points of $S$,*
(ii) *the restriction*
$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$
*is smooth,*
(iii) *$\mathscr{L}$ is $f$-ample, and*
(iv) *the fiber of $f$ over every geometric point of $U$ is a homogeneous space for a linear algebraic group, and the restriction of $\mathscr{L}$ to this fiber is a generator of the Picard group.*

*Then there exists a rational section of $f$.*

The more traditional formulation of Serre's Conjecture II is in terms of a non-algebraically closed field $K$, which here is the function field $k(S)$ of the surface $S$.

***Theorem 1.4***. – *Let $k$ be an algebraically closed field and let $K/k$ be the function field of a surface. Let $G$ be a connected, simply connected, semisimple algebraic group over $k$. Every $G$-torsor over $K$ is trivial.*

*In particular if $G_K$ is a simple algebraic group over $K$ of type $E_8$ then $G_K$ is itself split. Thus every $G_K$-torsor over $K$ is trivial. So Serre's Conjecture II holds over function fields $K$ for groups of type $E_8$.*

**Overview.** – Given a smooth, projective surface $S$ over an algebraically closed field $k$, there always exists a Lefschetz pencil of divisors on $S$. The generic fiber $C$ of this pencil is a smooth, projective, geometrically integral curve over the function field $\kappa = k(t)$. Given a projective, flat morphism $f : X \to S$ whose geometric generic fiber is integral and rationally connected, the fiber product $X_\kappa := C \times_S X$ is a projective $\kappa$-scheme together with a projective, flat morphism of $\kappa$-schemes $\pi : X_\kappa \to C$ whose geometric generic fiber is integral and rationally connected. Since the generic fiber of $\pi$ equals the generic fiber of $f$, rational sections of $f$ are really the same as rational sections of $\pi$. So it suffices to prove that $\pi$ has a section.

And the morphism $\pi$ has one advantage over $f$: the base change morphism

$$\pi \otimes \mathrm{Id} : X_\kappa \otimes_\kappa \overline{\kappa} \to C \otimes_\kappa \overline{\kappa}$$

does have a section by Theorem 1.2. By Grothendieck's work on the Hilbert scheme there exists a $\kappa$-scheme $\mathrm{Sections}(X/C/\kappa)$ parameterizing families of sections of $\pi$. The goal is to prove $\mathrm{Sections}(X/C/\kappa)$ has a $\kappa$-point. But we at least know it has a $\overline{\kappa}$-point. As with all Hilbert schemes, this is really a countable union of quasi-projective $\kappa$-schemes, $\sqcup_e \mathrm{Sections}^e(X/C/\kappa)$, where $\mathrm{Sections}^e(X/C/\kappa)$ is the open and closed subscheme parameterizing sections which have degree $e$ with respect to some $\pi$-relatively ample invertible sheaf $\mathscr{L}$.

The basic idea is to try to prove that $\mathrm{Sections}^e(X/C/\kappa)$ has some naturally defined closed $\kappa$-subscheme which is geometrically integral and geometrically rationally connected. Then we can apply Theorem 1.2 to this closed subscheme to produce a $\kappa$-point of $\mathrm{Sections}^e(X/C/\kappa)$, which is the same as a section of $\pi$.

Of course there is an obstruction to rational connectedness of $\mathrm{Sections}^e(X/C/\kappa)$: the Abel map

$$\alpha : \mathrm{Sections}^e(X/C/\kappa) \to \mathrm{Pic}^e_{C/\kappa}$$

sending each section of $\pi$ to the pullback of $\mathscr{L}$ by this section. Since there are no rational curves in the Abelian variety $\mathrm{Pic}^e_{C/\kappa}$, every rationally connected subvariety of $\mathrm{Sections}^e(X/C/\kappa)$ is contained in a fiber of $\alpha$. So the idea is to prove that for $e$ sufficiently positive, some irreducible component of the generic fiber of $\alpha$ is geometrically integral and geometrically rationally connected. Of course this is the same as proving that there exists an irreducible component $Z_e$ of $\mathrm{Sections}^e(X/C/\kappa)$ such that

$$\alpha|_{Z_e} : Z_e \to \mathrm{Pic}^e_{C/\kappa}$$

is dominant with integral and rationally connected geometric generic fiber. Observe that this would be enough to conclude the existence of a section of $\pi$: there are $\kappa$-points of $\mathrm{Pic}^e_{C/\kappa}$, e.g., coming from the basepoints of the Lefschetz pencil, and the fiber of $\alpha|_{Z_e}$ over these $\kappa$-points is then a geometrically integral and rationally connected variety defined over $\kappa = k(t)$. Such a variety has a $\kappa$-point by Theorem 1.2.

There are some issues. First of all if we change $\mathscr{L}$ then the Abel map $\alpha$ changes. For instance, if we replace $\mathscr{L}$ by $\mathscr{L}^{\otimes n}$ with $n > 1$, then the original Abel map is composed with the "multiplication by $n$" morphism on the Picard scheme. Because this is a finite map of degree $> 1$, the geometric generic fiber of the new Abel map will not be integral. So it is crucial to work with the correct invertible sheaf $\mathscr{L}$. If the geometric generic fiber of $f$ has Picard group isomorphic to $\mathbb{Z}$ (rationally connected varieties always have discrete Picard group), then this obstruction is equivalent to the well known *elementary obstruction* of Colliot-Thélène and Sansuc, cf. [**6**]. We impose vanishing of the elementary obstruction in a somewhat hidden manner through existence properties for "lines" in the generic fiber, i.e., curves of $\mathscr{L}$-degree 1. Observe that there are no curves of $\mathscr{L}^{\otimes n}$-degree 1, which indicates the connection with the elementary obstruction.

A second, weightier issue is that $\mathrm{Sections}^e(X/C/\kappa)$ typically is not proper. So it is extremely unlikely any interesting subvarieties are rationally connected. Fortunately it suffices to prove there is a component $Z_e$ as above for a compactification $\Sigma^e(X/C/\kappa)$ of $\mathrm{Sections}^e(X/C/\kappa)$. The compactification we use here comes from Kontsevich's moduli space of stable maps. But there is a third problem: this space will usually have more than one irreducible component. Some of these components have bad properties because the generic point parameterizes an obstructed section. So we restrict attention to those irreducible components which parameterize unobstructed sections, specifically what we call "$(g)$-free sections" where $g$ is the genus of $C$. Still there may be more than one irreducible component $Z$ parameterizing $(g)$-free sections.

We cannot fix this for any particular $\mathscr{L}$-degree, say $\epsilon$: for any particular integer $\epsilon$ there may well be more than one irreducible component $Z$ of $\Sigma^\epsilon(X/C/\kappa)$ parameterizing $(g)$-free sections. However the problem gets better as the $\mathscr{L}$-degree becomes more positive. There is a standard way of producing new sections from old: attach vertical rational curves to the section curve and deform this reducible curve to get an irreducible curve which is again a section. If the original section curve and vertical curves are sufficiently free, then the reducible curve does deform and the deformations are again unobstructed. In particular the new section is parameterized by a smooth point of $\Sigma^e(X/C/\kappa)$ for some $e > \epsilon$. Of course there are many ways of attaching and deforming, so we choose the simplest possible: attach vertical "lines", i.e., curves whose $\mathscr{L}$-degree equals 1. We use the somewhat colorful name "porcupine" to denote a reducible curve obtained from a $(g)$-free section by attaching free lines in fiber of $\pi$. Using these porcupines, we produce a sequence $(Z_e)_{e\geq\epsilon}$ of irreducible components $Z_e$ of $\Sigma^e(X/C/\kappa)$. Of course this presupposes the existence of many free lines to attach to our original section, and that leads to our first technical hypothesis: every point of