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by

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1. Introduction

This is a set of lecture notes for a course given at the Summer School "Uniformisation de familles de variétés complexes" organised by L. Meersseman in Dijon (France) August, 31-September 11, 2009 and funded by the ANR project "Complexe" (ANR-08-JCJC-0130-01), the École doctorale Carnot and the Institut Mathématique de Bourgogne.

These notes are meant to serve as an introduction to non abelian Hodge theory with a focus on its use in the Shafarevich problem.

Definitions and notations. – For background information on Kähler manifolds and Hodge Theory, a useful reference is [**60**]. We will not give any details on the facts and definitions already contained there.

In the sequel, X denotes a compact connected Kähler manifold and ω a Kähler form on X. Its universal covering space will be denoted by $\pi : \widetilde{X^{\mathrm{un}}} \to X$.

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Uniformization in several complex variables. Basic examples. – Uniformization in complex geometry aims at understanding the universal covering space, the fundamental group and hence the various covering spaces of complex manifolds. In these notes, I will focus on the compact Kähler case.

The Riemann uniformization theorem, whose first complete proof was given independently by Koebe and Poincaré in 1907, states that a simply connected Riemann surface is isomorphic to $\mathbf{P}^1(\mathbb{C})$, $\mathbb{C}$ or $\Delta = \{z \in \mathbb{C} | \; |z| < 1\}$. As a corollary, one can describe the universal covering space of a compact Riemann surface according to the following:

Theorem 1.0.1. – *Let C be a compact connected Riemann surface of genus g and $\pi : U \to C$ be its universal covering space.*

If $g = 0$: *then C is simply connected, $U = C$ and $C \simeq \mathbf{P}^1(\mathbb{C})$.*
If $g = 1$: *then $U \simeq \mathbb{C}$, $C \simeq \Lambda\backslash\mathbb{C}$ where $\Lambda \simeq \mathbb{Z}^2$ is a rank 2 discrete subgroup of $\mathbb{C}$.*
If $g \geq 2$: *then $U \simeq \Delta$, $C \simeq \Gamma\backslash\Delta$ where $\Gamma \subset PU(1,1)$ is a torsion-free cocompact discrete subgroup isomorphic to*

$$< a_1, b_1, \ldots, a_g, b_g \quad | \quad a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 >$$

It was realized early that the situation is much more complicated in several complex variables. For instance, the bidisk and the complex two ball are not isomorphic as complex manifolds but can be realized as the universal covering space of a complex projective surface of general type.

Compact Kähler manifolds with infinite fundamental groups are not too abundant. Nevertheless, a nice zoology of examples can be displayed.

Complex tori: Let $\Lambda \in \mathbb{C}^n$ be a rank $2n$ lattice. The complex torus $\Lambda\backslash\mathbb{C}^n$ is a compact Kähler manifold whose universal covering space is $\mathbb{C}^n$. It is a projective manifold (and hence an abelian variety) iff the weight -1 $\mathbb{Z}$-Hodge structure determined by $\Lambda \in \mathbb{C}^n$ can be polarized.

Hermitian locally symmetric spaces: Let Ω be a bounded symmetric domain (cf. [**45**]). Familiar examples of irreducible bounded symmetric domains are complex balls (aka complex hyperbolic spaces), Siegel upper half planes, period domains for K3 surfaces.

Let Γ be a cocompact torsion free lattice(1) in $\mathrm{Aut}(\Omega)$, then $\Gamma\backslash\Omega$ is a canonically polarized projective manifold whose universal covering space is Ω.

Kuga varieties: When the bounded symmetric domain Ω is classical, provided $\Gamma \in \mathrm{Aut}(\Omega)$ is well chosen (see [**45**] for details), then $\Gamma\backslash\Omega$ appears as a fine moduli space for a family of abelian varieties of genus g with non trivial $\mathbb{Q}$-endomorphism ring, polarization type and level structure. This gives rise to an abelian scheme $\pi : A_\Gamma \to \Gamma\backslash\Omega$. The universal covering of the smooth projective manifold A is then biholomorphic to $\Omega \times \mathbb{C}^g$.

(1) This always exists by a theorem of A. Borel.

Kodaira surfaces: A Kodaira surface is a projective surface S endowed with a smooth fibration $p : S \to C$ whose fibers have genus ≥ 2. Then the universal covering space can be realized as a bounded pseudoconvex domain in $\mathbb{C}^2$ ([**22**], Lemma 6.2, p.39).

Mostow-Siu surfaces and Deraux threefold: In [**40**], exotic (i.e. non complex hyperbolic) projective surfaces of negative curvature are constructed. Improving this construction, M. Deraux was able to produce an example in dimension 3 in [**11**]. In this examples, the universal covering space can be expressed as an infinite ramified covering space over a complex ball.

In all these cases, the universal covering space is Stein and contractible and the compact Kähler manifold is an Eilenberg- Mac Lane $K(\pi, 1)$.

Shafarevich conjecture on holomorphic convexity. – Further examples of compact Kähler manifolds with Stein universal covering space can easily be constructed. For instance, any smooth submanifold Y in a projective manifold X with Stein universal covering space has Stein universal covering space, too. On the other hand, even in case the universal covering space of X is contractible, the universal covering space of a sufficiently ample smooth complete intersection of dimension $d \geq 2$ has a non trivial π_d. In particular, its universal covering space is not contractible.

In this example, the manifold has a 1-connected holomorphic embedding into a compact Kähler $K(\pi, 1)$. Toledo's example of a compact Kähler manifold with non residually finite Kähler group comes equipped with an embedding into a smooth quasiprojective $K(\pi, 1)$ [**57**]. However, the fundamental groups of some projective manifolds constructed in [**12**] do not have a finite CW-complex as a $K(\pi, 1)$. In particular these manifolds cannot have a 1-connected holomorphic embedding into a compactifiable complex manifold with contractible universal covering space.

On the other hand, the universal covering space of a compact Kähler manifold may contain positive dimensional compact complex analytic subvarieties. The universal covering space of the blow-up at the origin of the complex torus $T = \Lambda \backslash \mathbb{C}^n$ is the blow up of $\mathbb{C}^n$ at all lattice points in Λ and contains an infinite collection of copies of $\mathbf{P}^{n-1}(\mathbb{C})$.

The Shafarevich conjecture takes these examples into account and predicts that the universal covering space of a complex projective manifold (compact and embeddable in $\mathbf{P}^N(\mathbb{C})$) should be holomorphically convex. [2] This problem is still open, in spite of the recent positive results contained in [**17**] [**18**].

Definition 1.0.2. – *A complex analytic space S is holomorphically convex if there is a proper holomorphic morphism $\pi : S \to T$ with $\pi_* \mathcal{O}_S = \mathcal{O}_T$ such that T is a Stein space. T is then called the* Cartan-Remmert reduction *of S.*

[2] This problem was actually not formulated as a conjecture in the last chapter in [**47**].

Remark 1.0.3. – *If S is a normal holomorphically convex complex analytic space (in particular a complex manifold), then its Cartan-Remmert reduction T is a normal Stein space.*

If S is holomorphically convex and $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in S escaping to infinity there exists a holomorphic function f such that

$$\lim_{n\to\infty} |f(x_n)| = +\infty.$$

Let us give some evidence towards the Shafarevich conjecture.

- If S is a compact Kähler surface with $\kappa(S) \leq 1$ then its universal covering space is holomorphically convex [**25**].
- If X_1 and X_2 have holomorphically convex universal coverings so has $X_1 \times X_2$.
- If X and Y are bimeromorphic compact Kähler manifolds, then X has a holomorphically convex universal covering iff Y has too.
- If $f : X \to Y$ is a holomorphic map such that $f_* : \pi_1(X) \to \pi_1(Y)$ has finite kernel and Y has a holomorphically convex universal covering then X has too.

In the last statement, one cannot drop the restriction on f_* since the universal covering space of a holomorphically convex manifold needs not be holomorphically convex[(3)].

This defect can be used as an excuse to consider a slightly more general problem. Let $H \subset \pi_1(X)$ be a normal subgroup. Say (X, H) satisfies (HC) iff $H\backslash\widetilde{X^{\mathrm{un}}}$ is holomorphically convex. Obviously, the Shafarevich conjecture states that $(X, \{e\})$ should satisfy (HC) if X is a complex projective manifold.

Lemma 1.0.4. – *If (X, H) satisfies (HC) and $f : Y \to X$ is an holomorphic map from a compact Kähler manifold then $(Y, f_*^{-1}H)$ satisfies (HC).*

If (X, H) satisfies (HC), then there is a proper holomorphic mapping with connected fibers

$$\widetilde{s^H} : H\backslash\widetilde{X^{\mathrm{un}}} \to \widetilde{S_H(X)}.$$

It contracts precisely the compact connected analytic subspaces of $H\backslash\widetilde{X^{\mathrm{un}}}$. The mapping s^H is equivariant under the Galois group $G = H\backslash\pi_1(X)$ which acts properly and cocompactly on $\widetilde{S_H(X)}$.

The quotient map $s^H : X \to G\backslash\widetilde{S_H(X)}$ is called the H-Shafarevich morphism. In the influential article [**33**], J. Kollár made it clear that constructing the H-Shafarevich morphism is the first step to settle when trying to prove (HC). The second step is to prove that the normal complex space $\widetilde{S_H(X)}$ is Stein—the problem can be reduced to constructing a strongly plurisubharmonic exhaustion function on $\widetilde{S_H(X)}$. In fact

(3) The minimal resolution of a small Stein neighborhood of an elliptic surface singularity is obviously holomorphically convex. The singularity can have a rational nodal curve as an exceptional divisor. This rational curve then unfolds in the universal covering space as a connected infinite chain of rational curves on which holomorphic functions are constant. Hence this covering space cannot be holomorphically convex.