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by

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1. Introduction

In 1985 D. Sullivan [**24**] had introduced a dictionary between two domains of complex dynamics: iterations of rational functions and Kleinian groups (both acting on the Riemann sphere). This dictionary motivated many remarkable results in both domains, starting from the famous Sullivan's no wandering domain theorem in the theory of rational iterations.

One of the principal objects used in the study of Kleinian groups is the hyperbolic 3- manifold associated to a Kleinian group, which is the quotient of its isometric action on the hyperbolic 3-space. M. Lyubich and Y. Minsky [**22**] have suggested to extend Sullivan's dictionary by providing an analogous construction for iterations of rational functions: hyperbolic laminations (with singularities). They are constructed from the natural homeomorphic extension of the dynamics: the backward orbit space (or natural extension). The latter space contains an infinite disjoint union of embedded copies of $\mathbb{C}$. Taking the latter union, strengthening its topology and completion yields what is called the Lyubich-Minsky affine orbifold lamination. Each its leaf carries an affine structure that is equivalent to either $\mathbb{C}$, or a quotient of $\mathbb{C}$ by a discrete group of affine Euclidean isometries. The lifted dynamics on the laminated space is leafwise-affine. Pasting a copy of the hyperbolic 3-space to every copy of $\mathbb{C}$ in the natural extension and applying similar strengthening and completion yields the Lyubich-Minsky hyperbolic orbifold lamination. Each its leaf is a complete hyperbolic 3-manifold (orbifold) that is isomorphic to either the hyperbolic 3-space $\mathbb{H}^3$, or its quotient by a discrete group of isometries fixing the infinity. Thus, the covering hyperbolic space $\mathbb{H}^3$ of each

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leaf has a canonical marked point "infinity" at its boundary. A horizontal plane (i.e., a plane parallel to the boundary) in the half-space model of $\mathbb{H}^3$ is called a *horosphere*. The isometries fixing the infinity transform the horospheres to the horospheres. This induces the horospheric lamination of the laminated space and its quotient by the lifted dynamics.

Recent studies of the hyperbolic 3-manifolds associated to Kleinian groups resulted in solutions of all big problems in the theory, first of all the Ahlfors' Measure Conjecture and Marden's Tameness Conjecture (see [**1, 6**] and references therein). There is a hope that studying the hyperbolic laminations associated to rational functions would shed a new light to the underlying dynamics. This is a completely open research area with few results.

We recall the background material and recent achievements in the theory of Kleinian groups and rational dynamics and present the state of art in the laminations associated to rational functions.

The arrangement of the horospheres in the above hyperbolic laminations is related to the behavior of the modules of the derivatives of the iterations of the rational function.

The vertical geodesic flow acts on $\mathbb{H}^3$ by translating points along the geodesics issued from the infinity. This induces the leafwise vertical geodesic flows acting on the hyperbolic orbifold lamination and its quotient. The horospheric laminations are the unstable laminations of the vertical geodesic flows. There is a hope that studying the vertical geodesic flow and the horospheric lamination would have applications in understanding the underlying dynamics. An application to the No Invariant Line Field Conjecture in a particular case will be presented in Section 5.

The classical results concerning the geodesic flows on compact hyperbolic surfaces say that the horocyclic lamination is minimal [**16**] and uniquely ergodic [**4, 10, 23**]. Their generalizations have found important applications in different domains of mathematics, including number theory. In Sections 5 and 6 we present their versions, proved in [**12, 13, 14**], for the quotient horospheric laminations (in the complement to the isolated 3-dimensional hyperbolic leaves) associated to appropriate rational functions. The latter include the hyperbolic functions and the postcritically-finite ones that do not belong to the following list of exceptions: powers $z^{\pm d}$, Chebyshev polynomials, Lattès examples. The latter list consists exactly of those rational functions for which the affine laminations are Euclidean: admit a continuous family of Euclidean metrics on the leaves [**19**]. One of the main results of [**12, 13**] (stated here as Theorem 5.14) says that for every rational function either the corresponding quotient horospheric lamination (in the complement to the isolated 3-dimensional hyperbolic leaves) is topologically transitive, or the affine lamination is Euclidean. One of the topological transitivity results stated here (Theorem 5.19) is new. Its proof, which is omitted here, is a minor modification of the argument from [**13**]. The main unique ergodicity results from [**14**] concern a class of minimal laminations that includes the minimal unstable laminations of Anosov diffeomorphisms and flows and also the quotient horospheric laminations associated to appropriate functions. The original proofs of the classical results on unique ergodicity use Markov partitions, K-property or harmonic analysis. The proof from [**14**] is an elementary geometrical argument that does not use them. We give its sketch in Section 6.

In Section 2 we recall the basic material about Kleinian groups and hyperbolic geometry. We present a brief historical survey of the Ahlfors' Measure Conjecture and the main idea of its proof: studying associated harmonic functions on the hyperbolic quotients. This idea is due to Ahflors and Thurston.

In Section 3 we recall the basic material on holomorphic dynamics and Sullivan's dictionary. Afterwards we introduce the natural extension (backward orbit space) and construct the Lyubich-Minsky affine and hyperbolic laminations associated to the hyperbolic rational functions and quadratic Chebyshev polynomial.

In Section 4 we construct the affine and hyperbolic laminations in the general case and discuss their basic topological properties. The total space of those laminations is not locally compact in general.

In Section 7 we present some open problems concerning Lyubich-Minsky laminations. Earlier lists of open problems may be found in [**22**], p. 80-83 and [**19**], Section 4.7.

2. Background material on Kleinian groups and hyperbolic geometry

2.1. Kleinian groups. – Recall that the *Möbius group* is the group $PSL_2(\mathbb{C})$ of the conformal automorphisms of the Riemann sphere.

Definition 2.1. – *A nontrivial Möbius transformation $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is parabolic, if it has a unique fixed point. Otherwise, it has exactly two fixed points. In the latter case there are two possible subcases:*

- hyperbolic transformation: the fixed points are attractor and repeller, the dynamics is north pole – south pole; or equivalently, the eigenvalues of the corresponding matrix from $SL_2(\mathbb{C})$ have distinct moduli;

- elliptic transformation: conformally-conjugated to a sphere rotation; or equivalently, the corresponding matrix has distinct eigenvalues with equal moduli.

Remark 2.2. – *Each parabolic transformation is conformally conjugated to a translation of $\mathbb{C}$. Each hyperbolic (elliptic) transformation is conformally conjugated to a linear transformation $\mathbb{C} \to \mathbb{C}$: $z \mapsto \lambda z$, $|\lambda| \neq 1$ (respectively, $|\lambda| = 1$).*

Definition 2.3. – *A Kleinian group is a finitely-generated discrete subgroup in $PSL_2(\mathbb{C})$. In addition, for simplicity we assume that all the Kleinian groups under question are torsion-free.*

Example 2.4. – *A cyclic group Γ_{hyp} generated by a hyperbolic transformation. A cyclic group Γ_{par} generated by a parabolic transformation. A group Γ_T of translations of $\mathbb{C}$ by vectors of a two-dimensional lattice.*

Example 2.5. – *A Fuchsian group: a discrete group of transformations in $PSL_2(\mathbb{C})$ preserving a circle (line) in $\overline{\mathbb{C}}$. A quasi-Fuchsian group: a discrete subgroup in $PSL_2(\mathbb{C})$ preserving a Jordan curve in $\overline{\mathbb{C}}$. In both cases the group preserves two complementary disks (simply-connected domains separated by the invariant Jordan curve). A (quasi-) Fuchsian group is called a surface group, if the quotient of each latter invariant domain is a compact Riemann surface.*

Example 2.6. – *A Schottky group. Fix four disjoint closed disks $B_1, \dots, B_4$ in $\overline{\mathbb{C}}$. Let $S_1, \dots, S_4$ denote their boundary circles. Consider two transformations $A, B \in PSL_2(\mathbb{C})$ acting as follows:*

- A sends S_1 to S_2, the exterior of B_1 to B_2;

- B sends S_3 to S_4, the exterior of B_3 to B_4.

It is well-known that the group $\Gamma =< A, B >$ (Schottky group with two generators) is free and discrete. Similar construction with arbitrary even number $2n$ of disks $B_1, \dots, B_{2n}$ yields a Schottky group with n generators.

Definition 2.7. – *The discontinuity set of a Kleinian group Γ is the maximal open subset $D_\Gamma \subset \overline{\mathbb{C}}$ with the following property: for every $x \in D_\Gamma$ there exists a neighborhood $U = U(x) \subset \overline{\mathbb{C}}$ such that $\gamma U \cap U = \varnothing$ for every $\gamma \in \Gamma \setminus 1$. The complement $L_\Gamma = \overline{\mathbb{C}} \setminus D_\Gamma$ is called the limit set of Γ.*

Remark 2.8. – *The sets D_Γ and L_Γ are Γ-invariant.*

Lemma 2.9. – *The limit set of every Kleinian group is the closure of the set of the fixed points of all its elements. Given arbitrary point $x \in \overline{\mathbb{C}}$, the limit set is the set of all the limit points of the orbit of x.*

***Theorem 2.10* (Ahlfors' finiteness theorem [2])**. – *The action $\Gamma : D_\Gamma \to D_\Gamma$ is proper discontinuous, and its quotient is a finite disjoint union of connected Riemann surfaces. Each of the latter is a finitely punctured compact Riemann surface.*

Example 2.11. – *The limit set of every cyclic group from Example 2.4 coincides with the set of fixed points of generator. The quotients D_Γ/Γ corresponding to the groups $\Gamma_{\rm hyp}$, $\Gamma_{\rm par}$ and Γ_T are respectively a complex torus, the cylinder $\mathbb{C}^* = \mathbb{C} \setminus 0 \simeq \mathbb{C}/\mathbb{Z}$ and a complex torus. The limit set of a (quasi-) Fuchsian surface group is the corresponding invariant circle (Jordan curve). The corresponding quotient of the discontinuity set is a union of two compact Riemann surfaces, which have "complex-conjugate" conformal types in the Fuchsian case. The limit set of a Schottky group is a Cantor set. The corresponding quotient of the discontinuity set is a compact Riemann surface, whose genus equals n: the number of the generators, i.e., the half of the number of the disks.*

Definition 2.12. – *A fundamental domain of a Kleinian group is an open subset $U \subset D_\Gamma$ (not necessarily connected) that is a fundamental domain for its action on D_Γ in the usual sense:*

- $\gamma U \cap U = \varnothing$ for every $\gamma \in \Gamma \setminus 1$;

- the union of the closures in D_Γ of the images γU is all of D_Γ.