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DEFORMATIONS OF HOLOMORPHIC AND TRANSVERSELY HOLOMORPHIC FOLIATIONS

by

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In a remarkable series of articles, published between 1957 and 1960, K. Kodaira and D.C. Spencer developed the theory of deformations of compact complex manifolds of arbitrary dimension.

This theory has important differences with respect to Teichmüller's theory. Whereas in complex dimension one almost complex structures are always integrable (as it is said by the theorem of existence of isothermal coordinates), in higher dimension the integrability condition is described by a first-order partial differential equation which is non-linear. By this reason the results in higher dimension are not so complete as in the case of Riemann surfaces.

The main differences are the following. On the one hand, the theory of deformations of a compact complex manifold M initiated by Kodaira and Spencer is a local theory. More precisely, it parametrizes up to isomorphism all the complex structures, on the underlying differentiable manifold, that are close enough to the given one by means of a (germ of) finite dimensional parameter space. On the other hand, this parameter space can be singular in contrast with Teichmüller's space which is always smooth.

In their approach, Kodaira and Spencer used power-series methods as well as the theory of elliptic differential operators. When one linearizes the problem of classifying complex structures close to a given one up to isomorphism, the space of solutions is naturally identified to the cohomology group $H^1(M, \Theta_M)$, where Θ_M is the sheaf of holomorphic vector fields on M. One can try to construct the solution of the non-linear problem as a certain function on $H^1(M, \Theta_M)$ given by a power series. At each step, the obstruction to add one more term to the series lies in $H^2(M, \Theta_M)$. If all the obstructions vanish, then one has to prove the convergence of the series so constructed.

By that method, Kodaira and Spencer proved the following two results:

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Rigidity theorem. – *If $H^1(M,\Theta_M)=0$ then the complex manifold M is rigid; that is, each complex structure M' close enough to M is isomorphic to M.*

Completeness theorem. – *If $H^2(M,\Theta_M)=0$ then a neighborhood of the origin in $H^1(M,\Theta_M)$ parametrizes, up to isomorphism, all the complex structures which are close enough to M.*

In 1962 M. Kuranishi was able to remove the hypothesis $H^2(M,\Theta_M)=0$ in the above theorem but allowing the parameter space to be a (singular) analytic space, although still keeping $H^1(M,\Theta_M)$ as its Zariski tangent space at the origin [**25**]. Here, we have to point out the talk on Kuranishi's theorem given by A. Douady at the Séminaire Bourbaki (cf. [**9**]). Using the theory of Banach analytic spaces Douady improved Kuranishi's result allowing the parameter spaces to be non-reduced and obtaining the most powerful version of the theorem.

Since then, Kodaira-Spencer's approach to deformation theory has been used to deal with different types of geometric structures on compact manifolds. Among them, holomorphic and transversely holomorphic foliations.

In [**10**] T. Duchamp and M. Kalka proved that the fundamental sheaf $\Theta^{\mathrm{tr}}_{\mathcal{F}}$ of a transversely holomorphic foliation $\mathcal{F}$ on a compact manifold has finite dimensional cohomology. (This result was also proved by X. Gómez-Mont by different methods in [**18**]). Moreover, these authors constructed an elliptic resolution of $\Theta^{\mathrm{tr}}_{\mathcal{F}}$, which mixes the de Rham with the Dolbeault complex, and used it to construct a Kuranishi space of deformations for transversely holomorphic foliations. However, their construction can only be carried out under a quite restrictive technical hypothesis. The general theorem on the existence of a Kuranishi space of deformations for an arbitrary transversely holomorphic foliation was obtained by J. Girbau, A. Haefliger and D. Sundararaman in [**15**]. These authors used a different resolution of the fundamental sheaf which is inspired in a previous paper by Kodaira and Spencer [**24**]. That article also develops the deformation theory of (non-singular) holomorphic foliations and discusses some important examples of holomorphic and transversely holomorphic foliations.

From a general point of view, the local theory of deformations of compact complex manifolds or of holomorphic or transversely holomorphic foliations on compact manifolds is already quite complete, although the computation of the Kuranishi spaces for concrete examples continues to be an active field of research. Nevertheless, there are important questions on the theory that remain largely open and that are receiving a renewed interest, like the study of deformations in the large or the deformations of complex foliations, as it is explained by L. Meersseman in other chapter of this monograph.

In these notes we try to explain the basic ideas of the deformation theory of compact complex manifolds as it was developed by Kodaira, Spencer and Kuranishi and then we show how this approach can be adapted to the deformation theory of holomorphic and transversely holomorphic foliations following the work of Girbau, Haefliger and Sundararaman. Our aim is to explain the philosophy of those theories, skipping the more technical results but emphasizing the main difficulties and the central ideas.

In the last Section we illustrate the theory with the discussion of some examples, showing different strategies to compute the Kuranishi space of some concrete complex manifolds or foliations. We end the notes with the statement of some open questions.

1. Complex structures and holomorphic foliations

All along these notes M will denote a smooth manifold of dimension m. We will be specially interested in the case in which M is closed, i.e. compact and without boundary. The word *smooth* has always to be understood in the sense of being of class C^∞.

If $m = 2n$, a complex structure τ on M is given by an atlas $\{U_i, \phi_i\}$ of M with the following two properties: (1) $\phi_i(U_i)$ is an open subset of $\mathbb{C}^n$ and (2) the corresponding cocycle $\{g_{ij}\}$, which is defined by the condition $\phi_j = g_{ji} \circ \phi_i$ where the identity has a sense, is given by holomorphic transformations. The manifold M, endowed with a complex structure τ, is called a complex manifold.

Let f be a smooth diffeomorphism of M and τ a complex structure defined by an atlas $\{U_i, \phi_i\}$. Clearly $\{f^{-1}(U_i), \psi_i = \phi_i \circ f\}$ defines a new complex structure $f^*\tau$ on M that we call the pull-back of τ by f. This new complex structure $f^*\tau$ is just a "coordinate change" of τ and the two structures will be thought as being equivalent. Hence a central question is the description of the quotient space

$$\mathcal{Z}(M) = \mathscr{C}(M)/\operatorname{Diff}(M),$$

where $\mathscr{C}(M)$ is the set of complex structures on M and $\operatorname{Diff}(M)$ denotes the group of smooth diffeomorphisms of M.

Examples 1.1. – (a) The complex affine space $\mathbb{C}^n$ and the complex projective space $\mathbb{C}P^n$ are complex manifolds.

(b) (Complex submanifolds) A smooth submanifold N of a complex manifold M is a complex manifold if N is locally defined by holomorphic submersions.

Notice that compact complex submanifolds of $\mathbb{C}^n$ are just finite sets. A theorem by Chow states that each complex submanifold N of $\mathbb{C}P^n$ is an algebraic projective variety, i.e. N is defined by a finite family of homogeneous polynomial equations $P_j(z_0, \dots, z_n) = 0$, where $(z_0, \dots, z_n)$ are the homogeneous coordinates of $\mathbb{C}P^n$.

(c) (Holomorphic quotients) Under certain conditions, if G is a group acting holomorphically on a complex manifold M, the quotient space M/G has a natural complex structure.

If G is a discrete group acting properly discontinuously on M then the quotient mapping $\pi\colon M \to M/G$ is a covering map and there is a unique complex structure on M/G for which π is holomorphic. Complex tori $\mathbb{T}^n = \mathbb{C}^n/\Lambda$, where Λ is a lattice of $\mathbb{C}^n$, are examples of quotient complex manifolds. Another class of examples is that of Hopf manifolds $W_f = \mathbb{C}^n - \{0\}/\langle f\rangle$, where f is an automorphism of $\mathbb{C}^n$ with the origin 0 as an attractive fixed point. Notice that W_f is diffeomorphic to $S^{2n-1} \times S^1$.

Let us consider the vector field ξ of $\mathbb{C}^2$ given by

$$\xi = z_1 \frac{\partial}{\partial z_1} + \lambda z_2 \frac{\partial}{\partial z_2} \tag{1}$$

where λ is a complex number fulfilling $\operatorname{Im}\lambda > 0$. For $i = 1, 2$ denote $L_i = \{z_i = 0\}$ and set $V = \mathbb{C}^2 - (L_1 \cup L_2)$. Then V is saturated by the holomorphic $\mathbb{C}$-action on $\mathbb{C}^2$ defined by the vector field ξ and it is easy to see that the quotient $E_\lambda = V/\mathbb{C}$ is naturally identified to an elliptic curve. Moreover, each elliptic curve can be obtained in that way by an appropriate choice of the parameter λ.

The above definition of a complex structure is essentially non-linear. Alternatively, the notion of complex structure can be defined as a linear object on M (a vector bundle) fulfilling an integrability condition. In this approach, it is the integrability condition which is non-linear.

A complex structure τ on M induces a decomposition

$$TM^c = T^{1,0} \oplus T^{0,1}$$

of the complexified tangent bundle $TM^c = TM \otimes \mathbb{C}$ of M as a direct sum of the complex subbundles $T^{1,0}$ and $T^{0,1}$ that, in terms of local coordinates $z_1, \ldots, z_n$, are defined as

$$T^{1,0} = \langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \rangle_{\mathbb{C}} \quad \text{and} \quad T^{0,1} = \langle \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \rangle_{\mathbb{C}}.$$

Note that $T^{0,1} = \overline{T^{1,0}}$ and that $T^{1,0}$ and $T^{0,1}$ are involutive subbundles of TM^c. This facts motivate the following definition.

Definition 1.2. – An *almost complex structure* on an even dimensional manifold M is given by a complex subbundle T of TM^c fulfilling $TM^c = T \oplus \overline{T}$. The almost complex structure is called *integrable* if the subbundle $\overline{T}$ is involutive, i.e. $[\overline{T}, \overline{T}] \subset \overline{T}$ (or, equivalently, $[T, T] \subset T$).

In an alternative way, an almost complex structure can be defined as an endomorphism $J : TM \to TM$ fulfilling $J^2 = -\operatorname{Id}$: the subbundle T corresponds to the eigenspace of eigenvalue i of the extension of J to TM^c.

An almost complex structure on M induces a complex structure on the (real) vector bundle TM in the following way. The composition of (real) vector bundles morphisms

$$TM \longrightarrow TM^c \longrightarrow T^{1,0},$$

where the first arrow is the natural inclusion and the second one is the natural projection with kernel $T^{0,1}$, is an isomorphism and the induced identification $TM \cong T^{1,0}$ provides TM with a complex structure.

We have seen that each complex structure induces an integrable almost complex structure. The following remarkable theorem states the equivalence mentioned above.