LANDAU DAMPING

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Panoramas et Synthèses

Numéro 38

2013

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique

Panoramas & Synthèses
38, 2013, p. 239–328

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by

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Abstract. – This course was taught in the summer of 2010 in the Centre International des Rencontres Mathématiques as part of a program on mathematical plasma physics related to the ITER project; it constitutes an introduction to the Landau damping phenomenon in the linearized and perturbative nonlinear regimes, following the recent work by Mouhot & Villani.

***Résumé* (Amortissement Landau).** – Ce cours a été enseigné à l'été 2010 au Centre International des Rencontres Mathématiques (CIRM), dans le cadre d'un programme sur l'étude mathématique des plasmas, en liaison avec le projet ITER; c'est une introduction au phénomène d'amortissement Landau en régime linéarisé et non-linéaire perturbatif, basé sur le travail récent de Mouhot & Villani.

Foreword

In 1936, Lev Landau devised the basic collisional kinetic model for plasma physics, now commonly called the Landau–Fokker–Planck equation. With this model he was introducing the notion of relaxation in plasma physics: relaxation *à la* Boltzmann, by increase of entropy, or equivalently loss of information.

In 1946, Landau came back to this field with an astonishing concept: relaxation without entropy increase, with preservation of information. The revolutionary idea that conservative phenomena may exhibit irreversible features has been extremely influential, and later led to the concept of violent relaxation.

This idea has also been controversial and intriguing, triggering hundreds of papers and many discussions. The basic model used by Landau was the linearized Vlasov–Poisson equation, which is only a formal approximation of the Vlasov–Poisson equation. In the present notes I shall present the recent work by Clément Mouhot and myself, extending Landau's results to the nonlinear Vlasov–Poisson equation in the

2010 Mathematics Subject Classification. – 82C99; 85A05, 82D10.
Key words and phrases. – Plasma physics, Landau damping, galactic dynamics, Vlasov–Poisson, plasma echo, Newton's methode, KAM Theory.

perturbative regime. Although this extension is still far from handling the mysterious fully nonlinear regime, it already turned out to be rich and tricky, from both the mathematical and the physical points of view.

These notes start with basic reminders about classical particle systems and Vlasov equations, assuming no prerequisite from modeling nor physics. Standard notation is used throughout the text, except maybe for the Fourier transforms: if $h = h(x, v)$ is a function on the position-velocity phase space, then $\widehat{h}$ stands for the Fourier transform in the x variable only, while $\widetilde{h}$ stands for the Fourier transform in both x and v variables. Precise conventions will be given later on.

A preliminary version of this course was taught in the summer of 2010 in Cotonou, Benin, on the invitation of Wilfrid Gangbo; it is a pleasure to thank the audience for their interest and enthusiasm. The first version of the notes was mostly typed during the nights of a meeting on wave turbulence organized by Christophe Josserand, in the welcoming library of the gorgeous Domaine des Treilles of the Fondation Schlumberger. Then the notes were polished as I was teaching the course, on the invitation of Éric Sonnendrücker, as part of the Cemracs 2010 program on plasma physics and mathematics of ITER, in the Centre International des Rencontres Mathématiques (CIRM), Luminy, near Marseille, France. I hope this text has retained a bit of the magical atmosphere of work and play which was in the air during that summer in Provence. The notes were later repolished and slightly increased on the occasion of a course in Université Claude Bernard (Lyon, France) in 2011, and after the constructive criticisms of an anonymous referee.

This foreword is also an opportunity to honor the memory of Naoufel Ben Abdallah, who tragically passed away, only days before the course in CIRM was held. Naoufel was a talented researcher, an energetic colleague, a reliable leader as well as a lively fellow. I cherish the memory of an astonishing hike which we did together, also with his wife Najla and our common friend Jean Dolbeault, in the Haleakala crater on Hawai'i, back in 1998. These memories of good times will not fade, and neither will the beauty of Naoufel's contribution to science.

1. Mean field approximation

The two main classes of kinetic equations are the collisional equations of Boltzmann type, modeling short-range interactions, and the mean field equations of Vlasov type, modeling long-range interactions. The distinction between short-range and long-range does not refer to the decay of the microscopic interaction, but to the fact that the relevant interaction takes place at distances which are much smaller than, or comparable to, the macroscopic scale; in fact both types of interaction may occur simultaneously. Collisional equations are discussed in my survey [**102**]. In this Chapter I will concisely present the archetypal mean field equations.

1.1. The Newton equations. – The collective interaction of a large population of "particles" arises in a number of physical situations. The basic model consists in the system of Newton equations in $\mathbb{R}^d$ (typically $d = 3$):

$$m_i\, \ddot{x}_i(t) = \sum_j F_{j\to i}(t), \tag{1.1}$$

where m_i is the mass of particle i, $x_i(t) \in \mathbb{R}^d$ its position at time t, $\ddot{x}_i(t)$ its acceleration, and $F_{j\to i}$ is the force exerted by particle j on particle i. Even if this model does not take into account quantum or relativistic effects, huge theoretical and practical problems remain dependent on its understanding.

The masses in (1.1) may differ by many orders of magnitude; actually this disparity of masses plays a key role in the study of the solar system, or the Kolmogorov–Arnold–Moser theory **[28]**, among other things. But it also often happens that the situation where all masses m_i are equal is relevant, at least qualitatively. In the sequel, I shall only consider this situation, so $m_i = m$ for all i.

If the interaction is translation invariant, it is often the case that the force derives from an **interaction potential**; that is, there is $W : \mathbb{R}^d \to \mathbb{R}$ such that

$$F = -\nabla W(x - y)$$

is the force exterted at position x by a particle located at position y. This formalism misses important classes of interaction such as magnetic forces, but it will be sufficient for our purposes.

Examples 1.1. – (a) $W(x-y) = \text{const.}\, \rho\, \rho'/|x-y|$ is the electrostatic interaction potential between particles with respective electric charges ρ and ρ', where $|x-y|$ is the Euclidean distance in $\mathbb{R}^3$; (b) $W(x-y) = -\text{const.}\, m\, m'/|x-y|$ is the gravitational interaction potential between particles with respective masses m and m', also in $\mathbb{R}^3$; (c) Essentially any potential W arises in some physical problem or the other, and even a smooth (or analytic!) interaction potential W leads to relevant and difficult problems.

As an example, let us write the basic equation governing the positions of stars in a galaxy:

$$\ddot{x}_i(t) = \mathcal{G} \sum_{j\neq i} m_j\, \frac{x_j - x_i}{|x_j - x_i|^3},$$

where $\mathcal{G}$ is the gravitational constant. Note that in this example, a star is considered as a "particle"! There are similar equations describing the behavior of ions and electrons in a plasma, involving the dielectric constant, mass and electric charges.

In the sequel, I will assume that all masses are equal and work in adimensional units, so masses will not explicitly appear in the equations.

But now there are as many equations as there are particles, and this means a lot. A galaxy may be made of $N \simeq 10^{13}$ stars, a plasma of $N \simeq 10^{20}$ particles... thus the equations are untractable in practice. Computer simulations, available on Internet, give a flavor of the rich and complex behavior displayed by large particle systems interacting through gravity. It is very difficult to say anything intelligent in front of these complex pictures!

This complex behavior is partly due to the fact that the gravitational potential is attractive and singular at the origin. But even for a smooth interaction W the large value of N would cause much trouble in the quantitative analysis. The **mean field limit** will lead to another model, more amenable to mathematical treatment.

1.2. Mean field limit. – The limit $N \to \infty$ allows to replace a very large number of simple equations by just one complicated equation. Although we are trading reassuring ordinary differential equations for dreaded partial differential equations, the result will be more tractable.

From the theoretical point of view, the mean field approximation is fundamental: not only because it establishes the basic limit equation, but also because it shows that the qualitative behavior of the system does not depend much on the exact value of the number of particles, and then, in numerical simulations for instance, we can replace trillions of particles by, say, millions or even thousands.

It is not a priori obvious how one can let the dimension of the phase space go to infinity. As a first step, let us double variables to convert the second-order Newton equations into a first-order system. So for each position variable x_i we introduce the velocity variable $v_i = \dot{x_i}$ (time-derivative of the position), so that the whole state of the system at time t is described by $(x_1, v_1), \ldots, (x_N, v_N)$. Let us write X^d for the d-dimensional space of positions, which may be $\mathbb{R}^d$, or a subset of $\mathbb{R}^d$, or the d-dimensional torus $\mathbb{T}^d$ if we are considering periodic data; then the space of velocities will be $\mathbb{R}^d$.

Since all particles are identical, we do not really care about the state of each particle individually: it is sufficient to know the state of the system *up to permutation of particles.* In slightly pedantic terms, we are taking the quotient of the phase space $(X^d \times \mathbb{R}^d)^N$ by the permutation group $\mathcal{S}_N$, thus obtaining a cloud of undistinguishable points.

There is a one-to-one correspondence between such a cloud $\mathscr{C} = \{(x_1, v_1), \ldots, (x_N, v_N)\}$ and the associated *empirical measure*

$$\widehat{\mu}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i, v_i)},$$

where $\delta_{(x,v)}$ is the Dirac mass in phase space at (x, v). From the physical point of view, the empirical measure counts particles in phase space.

Now the empirical measure $\widehat{\mu}^N$ belongs to the space $P(X^d \times \mathbb{R}^d)$, the space of probability measures on the single-particle phase space. This space is infinite-dimensional, but it is *independent of the number of particles.* So the plan is to re-express the Newton equations in terms of the empirical measure, and then pass to the limit as $N \to \infty$.

For simplicity I shall assume that X^d is either $\mathbb{R}^d$ or $\mathbb{T}^d$, and that the force derives from an interaction potential W. The following proposition establishes the link between the Newton equations and the empirical measure equation. Its formulation