ASYMPTOTIC-PRESERVING SCHEMES FOR FLUID MODELS OF PLASMAS

P. Degond, V. Grandgirard, Y. Sarazin,
S. C. Jardin, C. Villani

edited by

P. Degond

Panoramas et Synthèses

Numéro 39/40 2013

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique

Panoramas & Synthèses
39-40, 2013, p. 1–92

ASYMPTOTIC-PRESERVING SCHEMES FOR FLUID MODELS OF PLASMAS

by

P. Degond

Abstract. – These notes summarize a series of works related to the numerical approximation of plasma fluid problems. We construct so-called 'Asymptotic-Preserving' schemes which are valid for a large range of values (from very small to order unity) of the dimensionless parameters that appear in plasma fluid models. Specifically, we are interested in two parameters, the scaled Debye length which quantifies how close to quasi-neutrality the plasma is, and the scaled cyclotron period, which is inversely proportional to the magnetic field strength. We will largely focus on the ideas, in order to enable the reader to apply these concepts to other situations.

Résumé **(Schémas 'Asymptotic-Preserving' pour les modèles fluides de plasmas)**
Ces notes résument une série de travaux concernant l'approximation numérique de modèles fluides de plasmas. Nous construisons des schémas appelés 'Asymptotic-Preserving' qui sont valides pour une large plage de valeurs (depuis des valeurs très petites jusqu'a des valeurs d'ordre un) des paramètres adimensionnels qui apparaissent dans les modèles fluides de plasmas. Plus spécifiquement, nous nous intéressons à deux paramètres: la longueur de Debye adimensionnelle qui quantifie à quelle distance le plasma se trouve du régime quasineutre, et la période cyclotron adimensionnée, qui est inversement proportionnelle à l'intensité du champ magnétique. Nous nous focaliserons essentiellement sur les idées, de manière à donner les moyens au lecteur d'appliquer ces concepts à d'autres situations.

2010 Mathematics Subject Classification. – 82D10, 76W05, 76X05, 76N10, 76N20, 76L05.
Key words and phrases. – Plasma fluid models, Asymptotic-Preserving schemes, Debye length, cyclotron frequency, Mach number, quasi-neutrality, drift-fluid regime, law Mach-number limit, shock-capturing schemes, conservative schemes, implicit schemes, strongly anisotropic diffusion equations.

© Panoramas et Synthèses 39-40, SMF 2013

FOREWORD

The material of these notes is the product of a research programme which has extended over several years and has involved a large number of collaborations. I would like to address special thanks to my collaborators Fabrice Deluzet, Giacomo Dimarco, Alexei Lozinski, Marie-Hélène Vignal (Institut de Mathématiques de Toulouse), Stéphane Brull (Institut de Mathématiques de Bordeaux), N. Crouseilles, Eric Sonnedrücker (IRMA, Strasbourg), J-G. Liu (Duke University), C. Negulescu (LATP, Marseille) and my current or former students and post-docs, Céline Parzani, Pierre Crispel, Jacek Narski, Laurent Navoret, Alexandre Mouton, Dominique Savelief (Institut de Mathématiques de Toulouse), Sever Hirstoaga (IRMA, Strasbourg), Afeintou Sangam (Laboratoire J-A. Dieudonné, Nice), An-Bang Sun (X'ian University) and Min Tang (Laboratoire J-L. Lions, Paris).

The problems and questions which are dealt with in these notes have been strongly inspired by many years of collaboration and contracts with the Commissariat à l'Energie Atomique or the Centre National d'Etudes Spatiales, and I would like to thank Gérard Gallice, Jean Ovadia, Christian Tessieras (CEA-Cesta), Annalisa Ambroso (now at the company Areva), Anela Kumbaro, Pascal Omnes, Jacques Segré (CEA-Saclay), Gloria Falchetto, Xavier Garbet, Maurizio Ottaviani (CEA-Cadarache), Kim-Claire Le Than (CEA-DIF), Jacques Payan (CNES), as well as Franck Assous. I would also like to acknowledge support from the Fondation RTRA 'Sciences et Technologies Aéronautiques et Spatiales', under the 'Plasmax project', led by Florent Christophe (ONERA-Toulouse).

LECTURE I
INTRODUCTION

Numerical resolution of perturbation problems

These notes are about the application of the 'Asymptotic-Preserving' methodology to construct schemes for plasma fluid problem which sustain large variations of some of the characteristic dimensionless parameters of the plasma. We will be specifically concerned with two of these parameters, the scaled Debye length on the one hand and the scaled cyclotron period on the other hand. The former quantifies how close to quasi-neutrality the plasma is while the latter measures the confinement effects due to the magnetic field.

Let us consider a singular perturbation problem P^ε whose solutions converge to those of a limit problem P^0 when the perturbation parameter ε tends to zero. Usually, when $\varepsilon \ll 1$, standard numerical methods (like e.g., explicit methods in the case of time-dependent problems) break down. The reason is that the stability condition limits the allowed time step to a maximal value which depends on ε and tends to zero when $\varepsilon \to 0$. In the case of hyperbolic problems, this problem occurs if one of the wave-speeds tends to infinity with ε. For instance, in the low Mach-number limit of compressible fluids, the acoustic wave speeds tend to infinity when the Mach-number tends to zero.

In order to overcome such problems, the usual strategy consists in solving the limit problem P^0 instead of P^ε. For instance, in the case of the Low Mach-number limit, the incompressible Euler or Navier-Stokes equations will be solved. However, there are several difficulties with this strategy, which we now outline. The first one is that it supposes that P^0 has been previously determined and the second one is that it assumes that P^0 easy to solve. Both assumptions are by no means obvious. There are cases where the determination of the limit problem is difficult if not impossible. Even if P^0 is well-known, it usually involves equations of mixed type (for instance, the small Mach-number flow model involves a combination of equations of hyperbolic and elliptic character), where constraints (such as the divergence free constraint on the velocity) need to be enforced. The abundant literature on the Stokes and Navier-Stokes problem shows that enforcing this constraint is not an easy problem.

The problem becomes even more complex when the parameter ε is not uniformly small. This sentence may sound a little awkward, since ε is a number which should have a definite fixed value. However, ε is usually a ratio of characteristic lengths which may vary in space and time. For instance, in a plasma sheath, where the density drops by orders of magnitude, the value of the Debye length changes dramatically. In other cases, such as in boundary layers, one must change the scaling length from say the size of the experiment, to the typical dimension of the boundary layer. Therefore, the definition of a uniform value for ε is difficult, and it is more appropriate to view ε as a local quantity.

In such a situation, ε may be small in some areas and order unity in other regions. Then, the use of the limit problem P^0 leads to wrong results in the regions where ε is not small. To make a more accurate simulation, it is necessary to decompose the simulation domain into regions where $\varepsilon = \mathcal{O}(1)$ and regions where $\varepsilon \ll 1$ and to solve P^ε in the former and P^0 in the latter. However, the practical realization of this coupling strategy is very complex.

Indeed, first, it requires to define the location where the transition from P^ε to P^0 or vice-versa takes place. This is not an obvious question when ε has a smooth rather than abrupt transition. The results then depend on the particular location of this transition. This drawback can be slightly circumvented by the use of a smooth transition (like fictitious mixture models in multiphase flows for instance). Nonetheless, the need to designate a specific region where the shift between the two models takes place is detrimental to the robustness and the reliability of the model.

Once the location of the interface or transition region between P^ε and P^0 has been determined, the second problem to solve is the definition of the coupling strategy between the two models. Usually, P^0 involves some kind of reduction of information compared to P^ε. For instance, in the low-Mach number limit, the velocity field becomes divergence free, meaning that it depends on two independent scalar quantities instead of three like in compressible flows. At the interface, in the passage $P^\varepsilon \to P^0$, it is necessary to project the unknowns of P^ε onto those of P^0, and vice versa, in the passage $P^0 \to P^\varepsilon$ the unknowns of P^ε must be reconstructed from those of P^0. For instance, in the passage from compressible to low Mach-number regimes, the velocity must be projected onto a divergence free field. In the reverse transition, the irrotational part of the velocity must be reconstructed from a divergence free velocity.

The answer to such questions is by no means obvious and the simulation results also depend on the choices of the projection-reconstruction operators. Connection conditions can be sought by solving an interior layer problem connecting the state variables of the P^ε problem at $x = -\infty$ to those of the P^0 problem at $x = +\infty$ through a spatial rescaling of P^ε in the direction normal to the interface. However, quite often, this analysis does not lead to a closed set of connection conditions. The interior layer problem itself carries some approximations because it involves a rescaling which leads to neglecting all derivatives in the tangential direction to the interface.

Finally, supposing that the questions above have found a satisfactory answer, the whole strategy must be practically implemented. The mesh must be constrained to match the interface. Therefore, if the interface is not planar, which is very likely in a realistic case, an unstructured mesh must be used. Additionally, the interface location may have to evolve in time. This brings several additional questions. First, the motion strategy for the interface must be defined. What criterion will decide for this motion ? Second, if the interface is moved, the mesh must be moved accordingly, otherwise the matching with the interface will be lost. Moving meshes with time is complex, uneasy and costly, both in terms of CPU time and of development time.

All these questions have led an increasing number of teams to look for schemes which are valid in both the $\varepsilon = \mathcal{O}(1)$ and $\varepsilon \ll 1$ regimes. For instance, there is an increasing literature about so-called 'all-speed schemes' for compressible flows, which are valid for all values of the Mach number [**5, 26, 32, 46**]. However, some precautions must be made because, a scheme may be stable in the $\varepsilon \ll 1$ limit and yet provide a wrong solution, i.e., a solution which is not consistent with the P^0 problem. The correct concept for doing so is the so-called Asymptotic-Preserving (AP) scheme, which is described in the next section.

Asymptotic-Preserving methodology

The concept of an Asymptotic-Preserving method has been introduced by S. Jin [**48**] for transport in diffusive regimes. A scheme $P^{\varepsilon,h}$ for P^ε with discretization parameters h (standing for both the time and space steps) is called Asymptotic Preserving (or AP) if it is stable irrespective of how small the perturbation parameter ε is, and if it leads to a scheme $P^{0,h}$ which is consistent with the limit problem P^0 when ε tends to zero with fixed discretization parameters h. This property is illustrated in the commutative diagram below.