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BOUNDARY MOTIVE, RELATIVE MOTIVES AND EXTENSIONS OF MOTIVES

by

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Abstract. – We explain the role of the boundary motive in the construction of certain Chow motives, and of extensions of Chow motives. Our two main examples concern proper, singular surfaces and fibre products of a universal elliptic curve.

0. Introduction

This article contains largely extended notes of a short series of lectures delivered during the *École d'été franco-asiatique "Autour des motifs"*, which took place at the IHÉS in July 2006. The task which I was assigned was to explain the role of the *boundary motive*, and I hope that the present article will make a modest contribution to this effect.

By definition **[21]**, the boundary motive $\partial M(X)$ of a variety X over a perfect field k fits into a canonical exact triangle

$$(*) \qquad \partial M(X) \longrightarrow M(X) \longrightarrow M^c(X) \longrightarrow \partial M(X)[1]$$

in the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of effective *geometrical motives*. This triangle establishes the relation of the boundary motive to $M(X)$ and $M^c(X)$, the *motive* of X and its *motive with compact support*, respectively **[19]**.

One way to explain its interest is to start with the notion of extensions. Indeed, most of the existing attempts to prove the Beilinson or Bloch–Kato conjectures on special values of L-functions necessitate the construction of extensions of (Chow) motives, and the explicit control of their realizations (Betti, de Rham, étale...). Often, the

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source of these extensions is *localization*, which expresses the motive with compact support of a non-compact variety X as an extension of the motive of a compactification X^* by the motive of the complement $X^* - X$. The realizations of these extensions then correspond to cohomology with compact support of X. This approach is clearly present e.g. in Harder's work on special values [**12**].

Thus, given two Chow motives, one may try to use localization to construct an extension of one by the other. Here, we base ourselves on the principle that the given Chow motives are "basic", and that the extension is "difficult" to obtain. But one may also invert the logic: given a "mixed" motive, try to use localization to construct the Chow motives used to build it up; let us refer to this problem as "resolution of extensions".

The purpose of this article is to establish that the boundary motive plays a role both for the construction and for the resolution of extensions *via* localization. In Section 1, we start by making precise the relation between localization and the boundary motive. In fact, the triangle $(*)$ turns out to be obtained by "splicing" the localization triangle and its dual. We chose to discuss this relation first in the Hodge theoretic realization, and in the special case of a complement X of two points in an elliptic curve over $\mathbb{C}$ (Examples 1.1, 1.3 and 1.5), and deduce from that discussion the general picture in Hodge theory (Theorems 1.6 and 1.7), concerning compactifications of a fixed variety X over $\mathbb{C}$. We observe in particular (Corollary 1.8) that when X is smooth, then any smooth compactification induces a *weight filtration* on the boundary cohomology of X, i.e., on the Hodge realization of the boundary motive.

In order to formulate the motivic analogues of these results, we need the right notion of weights for motives. It turns out that this notion is given by *weight structures*, as recently introduced and studied by Bondarko [**4**]. We review the definition, and the basic properties of weight structures, including their application to motives (Theorem 1.11): according to Bondarko, there is a canonical such structure on the triangulated category $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$, and its *heart* equals the category $\mathrm{CHM}^{\mathrm{eff}}(k)$ of effective Chow motives. The motivic analogue of Corollary 1.8 holds: according to Corollary 1.16, any smooth compactification of a fixed variety X which is smooth over k induces a weight filtration on $\partial M(X)$.

Then we try to invert this process (hoping for this inversion to allow us to resolve extensions). The precise statement is given in Theorem 1.18, which states that for fixed X, there is a canonical bijective correspondence (discussed at length in Construction 1.17) between isomorphism classes of two types of objects: (1) weight filtrations on $\partial M(X)$, and (2) certain effective Chow motives M_0 through which the morphism $M(X) \to M^c(X)$ factors. An analogous statement (Variant 1.23) holds for direct factors of $\partial M(X)$, $M(X)$, and $M^c(X)$, provided that they are images of an idempotent endomorphism of the whole exact triangle $(*)$. In this correspondence, the passage to isomorphism classes cannot be avoided because of the necessity to *choose* cones of certain morphisms in the triangulated category $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. This causes (at least) one important problem, namely the lack of functoriality of the representatives of the isomorphism classes. In order to obtain functoriality, Construction 1.17 thus needs to be rigidified.

In the rest of Section 1, we describe the approach from [**23**] to rigidification, hence functoriality. It is based on the notion of motives *avoiding certain weights*. If a direct factor $\partial M(X)^e$ of $\partial M(X)$ is without weights -1 and 0, then an effective Chow motive M_0 is canonically and functorially defined (Complement 1.24). Given the nature of the realizations of M_0, it is natural to call it the *e-part of the interior motive* of X. Its main properties are established in [**23**, Sect. 4]. Note however (Problem 1.22) that the above condition on absence of weights is never satisfied for the whole of $\partial M(X)$— unless $\partial M(X) = 0$. In order to make this approach work, we thus need an idempotent endomorphism e of the exact triangle $(*)$, giving rise to a direct factor

$$\partial M(X)^e \longrightarrow M(X)^e \longrightarrow M^c(X)^e \longrightarrow \partial M(X)^e[1] \, .$$

Section 2 shows how the theory of *smooth relative Chow motives* can be employed to construct endomorphisms of the exact triangle $(*)$. Fix a base scheme S, which is smooth over k. Theorem 2.2 establishes the existence of a functor from the category of smooth relative Chow motives over S to the category of exact triangles in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. On objects, it is given by mapping a proper, smooth S-scheme X to the exact triangle

$$\partial M(X) \longrightarrow M(X) \longrightarrow M^c(X) \longrightarrow \partial M(X)[1] \, .$$

We should mention that as far as the $M(X)$-component is concerned, the functoriality statement from Theorem 2.2 is just a special feature of results by Déglise [**9**], Cisinski–Déglise [**7**] and Levine [**13**] (see Remarks 2.3 and 2.13 for details). However, the application of the results from [loc. cit.] to the functor ∂M is not obvious. This is one of the reasons why we follow an alternative approach. It is based on a relative version of *moving cycles* [**21**, Thm. 6.14]. This also explains why we are forced to suppose the base field k to admit a strict version of resolution of singularities. Theorem 2.5 and Corollary 2.15 then analyze the behaviour of the functor from Theorem 2.2 under change of the base S. Another reason for us to choose a cycle theoretic approach was that it becomes then easier to keep track of the correspondences on $X \times_k X$ commuting with our constructions. Our main application (Example 2.16) thus concerns correspondences "of Hecke type" yielding endomorphisms of the exact triangle $(*)$.

In Section 3, we apply these principles to Abelian schemes. More precisely, the main result of [**10**] on the Chow–Künneth decomposition of the relative motive of an Abelian scheme A over S (recalled in Theorem 3.1) yields canonical projectors in the relative Chow group. Given our analysis from Section 2, it follows that they act idempotently on the exact triangle

$$\partial M(A) \longrightarrow M(A) \longrightarrow M^c(A) \longrightarrow \partial M(A)[1] \, .$$

In Sections 4 and 5, we discuss two examples. Section 4 concerns normal, proper surfaces X^*. We first recall the basic construction of the *intersection motive* $M^{!*}(X^*)$ of X^*, following previous work of Cataldo and Migliorini [**6**], and review some of the material from [**25**]. In particular (Proposition 4.3), we recall that $M^{!*}(X^*)$ is co- and contravariantly functorial under finite morphisms of proper surfaces. We then analyze the precise relation to the weight filtration of the boundary motive of a dense, open sub-scheme $X \subset X^*$, which is smooth over k (Theorem 4.4), following the lines of Construction 1.17. We finish the section with a discussion of the case of Baily–Borel compactifications of Hilbert surfaces. We recall, following [**25**, Sect. 6 and 7], that localization allows to construct non-trivial extensions of a certain Artin motive by a direct factor of $M^{!*}(X^*)$. Using Proposition 4.3, we then establish stability of $M^{!*}(X^*)$ under the correspondences "of Hecke type" constructed in Example 2.16.

In Section 5, we discuss fibre products of the universal elliptic curve over the modular curve of level $n \geq 3$. We review some of the material from [**16**] and [**23**, Sect. 3 and 4]. Notably (Proposition 5.3), we recall that in this geometrical setting, the condition from Complement 1.24 on the absence of weights -1 and 0 in the boundary motive is satisfied. Thus, the interior motive can be defined. The new ingredient is Example 5.4, where we use rigidity of our construction to give a proof "avoiding compactifications" of equivariance of the interior motive under the correspondences "of Hecke type".

As mentioned above, this article is primarily intended to be a general introduction to the construction and to the applications of boundary motives. For many details of the proofs, we shall refer to our earlier articles [**21**] and [**23**]. Let us however indicate that various parts of this paper discuss original constructions. This is true in particular for Section 2 (on relative motives and functoriality), including our study of Hecke equivariance. We expect these constructions to be of interest in other contexts than those discussed in Sections 4 and 5.

For further developments of the theory of boundary motives and their applications to special classes of algebraic varieties and to their associated motives, in particular to the motives of Shimura varieties, we refer also to [**22, 24**].

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Notation and conventions. – k denotes a fixed perfect base field, Sch/k the category of separated schemes of finite type over k, and $\mathrm{Sm}/k \subset \mathrm{Sch}/k$ the full sub-category of objects which are smooth over k. When we assume k to admit resolution of singularities, then it will be in the sense of [**11**, Def. 3.4]: (i) for any $X \in \mathrm{Sch}/k$, there exists an abstract blow-up $Y \to X$ [**11**, Def. 3.1] whose source Y is in Sm/k, (ii) for any $X, Y \in \mathrm{Sm}/k$, and any abstract blow-up $q : Y \to X$, there exists a sequence of blow-ups $p : X_n \to \ldots \to X_1 = X$ with smooth centers, such that p factors through q. We say that k admits strict resolution of singularities, if in (i), for any given dense open subset U of the smooth locus of X, the blow-up $q : Y \to X$ can be chosen to