# The full faithfulness conjectures in characteristic $p$

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# THE FULL FAITHFULNESS CONJECTURES IN CHARACTERISTIC $p$

*by*

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***Abstract.*** – We present a triangulated version of the conjectures of Tate and Beilinson on algebraic cycles over a finite field. This sheds a new light on Lichtenbaum's Weil-étale cohomology.

## Introduction

It is generally understood that the "standard" conjectures on mixed motives predict that certain triangulated realisation functors should be conservative. The aim of this text is to explain that, at least in characteristic $p$, they predict much more: namely, that suitable triangulated realisation functors should be *fully faihtful.*

The main result is the following. Let $\mathbf{F}$ be a finite field, and let $\mathrm{DM}_{\text{ét}}(\mathbf{F})$ be Voevodsky's stable category of (unbounded, étale) motivic complexes. It contains the category $\mathrm{DM}_{\text{ét}}^{\text{eff}}(\mathbf{F})$ of effective motivic complexes as a full subcategory. Let $l$ be a prime number different from $\operatorname{char}\mathbf{F}$. By work of Ayoub [**6**], there is a pair of adjoint functors:

$$\mathrm{DM}_{\text{ét}}(\mathbf{F}) \underset{R_l}{\overset{\Omega_l}{\leftrightarrows}} \hat{D}_{\text{ét}}(\mathbf{F}, \mathbf{Z}_l)$$

where the right hand side is Ekedahl's category of $l$-adic coefficients [**24**]. In particular, we have the object

$$\Gamma = \Omega_l(\mathbf{Z}_l) \in \mathrm{DM}_{\text{ét}}(\mathbf{F}).$$

***Theorem 1* (cf. Corollary 9.8.4).** – *The following conditions are equivalent:*

(i) *The Tate conjecture (on the poles of the zeta function) and the Beilinson conjecture (on rational equivalence agreeing with numerical equivalence) hold for any smooth projective $\mathbf{F}$-variety.*

(ii) $\Gamma \in \mathrm{DM}_{\text{ét}}^{\text{eff}}(\mathbf{F})$.

The full faithfulness statement announced above appears as another equivalent condition in Proposition 10.3.3 a); further equivalent conditions (finite generation of Hom groups) appear in Theorem 10.4.2. After the fact, see §11, these reformulations involve Weil étale cohomology. For a case when they hold in the triangulated context, see Theorem 12.2.1.

Curiously, the Beilinson conjecture and the Parshin conjecture (on vanishing of higher rational $K$-groups of smooth projective $\mathbf{F}$-varieties) are sufficient to imply the existence of a motivic $t$-structure on $\mathrm{DM}_{\mathrm{gm}}(\mathbf{F},\mathbf{Q})$, see Proposition 10.5.1 as well as Remark 10.5.2 (1).

A problem is that there is no known analogue of this picture in characteristic 0 at the moment. While in characteristic $p$ a single $l$-adic cohomology is sufficient to approach cycles modulo rational equivalence, it seems that in characteristic 0 one should consider the full array of realisation functors, plus their comparison isomorphisms. Even with this idea it does not seem obvious how to get a clean conjectural statement. In the light of §11, this might be of great interest to get the right definition of Weil-étale cohomology in characteristic 0.

This is a write-up of the talk I gave at the summer school on July 27, 2006. Much of the oral version was tentative because the suitable $l$-adic realisation functors were not constructed at the time. The final version is much more substantial than I had envisioned: this is both because of technical difficulties and because I tried to make the exposition as pedagogical as possible, in the spirit of the summer school. I hope the reader will bear with the first reason, and be satisfied with the second one.

I also hope that some readers will, like me, find the coherence and beauty of the picture below compelling reasons to believe in these conjectures.

I wish to thank Joseph Ayoub for a great number of exchanges while preparing this work, and the referee for a thorough reading which helped me improve the exposition.

**Notation.** – $k$ denotes a perfect field; we write $\mathrm{Sm}(k)$ for the category of smooth separated $k$-schemes of finite type. When $k$ is finite we write $\mathbf{F}$ instead of $k$ and denote by $G \simeq \hat{\mathbf{Z}}$ its absolute Galois group.

If $\mathscr{C}$ is a category, we write $\mathscr{C}(X,Y)$, $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ or $\mathrm{Hom}(X,Y)$ for the set of morphisms between two objects $X, Y$, according to notational convenience.

## 1. General overview

This section gives a background to the sequel of the paper.

**1.1. Triangulated categories of motives.** – As explained in André's book [**3**, Ch. 7], the classical conjectures of Hodge and Tate, and less classical ones of Grothendieck and Ogus, may be interpreted as requesting certain realisation functors on pure Grothendieck motives to be fully faithful. These conjectures concern algebraic cycles on smooth projective varieties modulo homological equivalence. On the other hand, both Bloch's answer to Mumford's nonrepresentability theorem for 0-cycles [**12**, Lect. 1] and Beilinson's approach to special values of $L$-functions [**7, 9**] led to conjectures on cycles modulo *rational* equivalence: the conjectures of Bloch-Beilinson and Murre (see Jannsen [**48**] for an exposition).

This development came parallel to another idea of Beilinson: in order to construct the (still conjectural) abelian category $\mathscr{M}(k)$ of mixed motives over a field $k$, one might start with the easier problem of constructing a triangulated category of motives, leaving for later the issue of finding a good $t$-structure on this category. Perhaps Beilinson had two main insights: first, the theory of perverse sheaves he had been developing with Bernstein, Deligne and Gabber [**10**] and second, his vanishing conjecture for Adams eigenspaces on algebraic $K$-groups (found independently by Soulé which deals with an *a priori* obstruction to the existence of $\mathscr{M}(k)$.

The latter programme: constructing triangulated categories of motives, was successfully developed by Levine [**57**], Hanamura [**32, 34, 33**] and Voevodsky [**82**] independently. All three defined tensor triangulated categories of motives over $k$, by approaches similar in flavour but quite different in detail. It is now known that all these categories are equivalent, if $\operatorname{char} k = 0$ or if we take rational coefficients.(1) More precisely, the comparison between Levine's and Voevodsky's categories is due to Levine in characteristic 0 [**57**, Part I, Ch. VI, 2.5.5] and to Ivorra in general [**41**], while the comparison between Hanamura's and Voevodsky's categories is due to Bondarko [**14**] and independently to Hanamura (unpublished).

These three constructions extend when replacing the field $k$ by a rather general base $S$ [**57, 86, 35**](2). At this stage, the issue of Grothendieck's six operations [**9**, 5.10 A] starts to make sense. In a talk at the ICTP in 2002, Voevodsky gave hints on how to carry this over in an abstract framework which would fit with his constructions, at least for the four functors $f^*$, $f_*$, $f_!$ and $f^!$. This programme was taken up by Ayoub in [**4**]; he added a great deal to Voevodsky's outline, namely a study of the missing operations $\otimes$ and $\underline{\mathrm{Hom}}$ plus related issues like constructibility and Verdier duality, as well as an impressive theory of specialisation systems, a vast generalisation of the theory of nearby cycle functors.

It remained to see whether this abstract framework applied to categories of motives over a base, for example to the Voevodsky version $S \mapsto \mathrm{DM}(S)$ constructed using relative cycles ("sheaves with transfers"). It did apply to a variant "without transfers" $S \mapsto \mathrm{DA}(S)$ (as well as to Voevodsky's motivating example: the Morel-Voevodsky

(1) Gabber's recent refinement of de Jong's alteration theorem [**39**] now allows us to just invert the exponential characteristic for these theorems.

(2) As far as I know, no comparison between these extensions has been attempted yet.

stable $\mathbf{A}^1$-homotopy categories $S \mapsto \mathrm{SH}_{\mathbf{A}^1}(S)$): see [**4**, Ch. 4] for this. It did not apply directly to DM, however. This issue was solved to some extent by Cisinski and Déglise [**17**], who showed that the natural functor $\mathrm{DA}_{\text{ét}}(S,\mathbf{Q}) \to \mathrm{DM}(S,\mathbf{Q})$ is an equivalence of categories when $S$ is a normal scheme, where $\mathrm{DA}_{\text{ét}}$ is an étale variant of DA. All this will be explained in much more detail in §6.

**1.2. Motivic conjectures and categories of motives.** – It is both a conceptual and a tactical issue to reformulate the conjectures alluded to at the beginning of §1.1 in this triangulated framework. The first necessary thing is to have triangulated realisation functors at hand. In Levine's framework, many of them are constructed in his book [**57**, Part I, Ch. V]. In Voevodsky's framework, with rational coefficients and over a field of characteristic 0, this was done by Huber using her triangulated category of mixed realisations as a target [**36**]. Over a separated Noetherian base, with integral coefficients and for $l$-adic cohomology, this was done by Ivorra [**40, 42**].

Then came up the issue whether realisation functors commute with the six operations. The only context where the question made full sense was Ivorra's. But there were three problems at the outset: Ivorra's functors 1) are only defined on geometric motives, and 2) are contravariant. The third problem is that the formalism of the six operations is not known to exist on $S \mapsto \mathrm{DM}(S)$ in full generality, as explained above.

These issues were recently solved by Ayoub who constructed covariant $l$-adic realisation functors from $\mathrm{DA}_{\text{ét}}(S)$ to Ekedahl's $l$-adic categories $\hat{D}(S,\mathbf{Z}_l)$ [**6**]. He proved that they commute with the six operations and with the right choice of a specialisation system. More details are in §6.

## 2. The Tate conjecture: a review

In this section, $\mathbf{F} = \mathbf{F}_q$ is a finite field with $q$ elements. The main reference here is Tate's survey [**80**].

**2.1. The zeta function and the Weil conjectures.** – Let $X$ be an $\mathbf{F}$-scheme of finite type. It has a zeta function:

$$\zeta(X,\mathrm{s}) = \exp\left(\sum_{n\geq 1} |X(\mathbf{F}_{q^n})| \frac{q^{-ns}}{n}\right) = \prod_{x\in X_{(0)}} (1-|\mathbf{F}(x)|^{-s})^{-1}$$

Weil conjectured that $\zeta(X,s) \in \mathbf{Q}(q^{-s})$ for any $X$: Dwork was first to prove it in [**23**]. A different proof, based on $l$-adic cohomology, was given by Grothendieck et al in [**2**]. It provided the following extra property, also conjectured by Weil: if $X$ is smooth projective, there is a functional equation of the form

$$\zeta(X,\mathrm{s}) = AB^s\zeta(X, \dim X - s)$$

where $A, B$ are constants.