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edited by

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Panoramas et Synthèses

Numéro 44

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique

Panoramas & Synthèses
44, 2014, p. 1–61

WAVE PROPAGATION AND IMAGING IN RANDOM WAVEGUIDES

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Abstract. – We present a rigorous asymptotic analysis of wave propagation in waveguides with random boundaries and with random fluctuations of the wave speed. We consider scalar waves in both two and three dimensions. The asymptotic analysis is with respect to the small magnitude of the random fluctuations, as they occur naturally in applications like underwater acoustics. Cumulative scattering effects of such fluctuations become significant at long distances of propagation. They couple the waveguide modes and cause loss of coherence of the wave field. We quantify explicitly mode coupling and loss of coherence via three important length scales: the scattering mean free path, the transport mean free path, and the equipartition distance. Knowledge of these scales is important for improving imaging methods and understanding their limitations. We consider a source imaging problem in a random waveguide. The data are the signals recorded by a receiver array far from the source. We study coherent array imaging and contrast it with the time-reversal process. We show that scattering by the random inhomogeneities is beneficial in general for the time reversal process, but it always impedes imaging. The resolution and robustness of the images deteriorate as the waves travel farther in the waveguide from the source to the array. At distances that exceed the scattering mean free path of all the modes, the wave field becomes incoherent and imaging can only be done with parametric estimation methods based on models of transport of energy. We analyze in detail the degradation of images in random waveguides, and illustrate the results with numerical simulations.

Résumé. – Nous présentons une analyse asymptotique rigoureuse de la propagation des ondes dans des guides d'ondes avec des bords aléatoires et avec des fluctuations aléatoires de la vitesse de propagation. Nous considérons des ondes scalaires en deux et trois dimensions. L'analyse est asymptotique par rapport à la faible amplitude des fluctuations aléatoires, comme il se produit naturellement dans des applications comme l'acoustique sous-marine. Les effets cumulatifs de diffusion dues à ces fluctuations deviennent importants sur de longues distances de propagation. Ils se manifestent par un couplage des modes du guide et une perte de cohérence du champ d'onde. Nous quantifions explicitement le couplage de modes et de la perte de cohérence par l'intermédiaire de trois échelles de longueur importantes : le libre parcours

2010 Mathematics Subject Classification. – 76B15, 35R60, 60F05, 35R30.
Key words and phrases. – Imaging, wave propagation, random media, asymptotic analysis.

moyen de diffusion, le libre parcours moyen de transport et la distance d'équipartition. La connaissance de ces échelles de longueur est importante pour améliorer les méthodes d'imagerie et pour comprendre leurs limites. Nous étudions un problème d'imagerie de source dans un guide d'ondes aléatoire. Les données sont les signaux enregistrés par un réseau de récepteurs situés loin de la source. Nous considérons un procédé d'imagerie cohérente par migration et nous la comparons avec le processus de retournement temporel. Nous montrons que la diffusion par les inhomogénéités aléatoires est bénéfique en général pour le processus de retournement temporel, mais il détériore toujours l'imagerie. La résolution et la robustesse des images se dégradent au fur et à mesure que la distance entre la source et le réseau de récepteurs augmente. A des distances supérieures au libre parcours moyen de diffusion de l'ensemble des modes, le champ d'ondes est incohérent et l'imagerie ne peut être effectuée par des méthodes d'estimation paramétrique basées sur des modèles de transport d'énergie. Nous analysons en détail la dégradation des images dans des guides d'ondes aléatoires, et illustrons les résultats par des simulations numériques.

1. Introduction

Array imaging is an important technology with a wide range of applications in underwater acoustics, seismology, non-destructive evaluation of materials, medical ultrasound, and elsewhere. It is concerned with locating remote, compactly supported sources and/or scatterers from measurements at arrays of sensors. The sensors are devices that transform one form of energy into another. Depending on the application they may be antennas that convert electromagnetic waves to/from electric signals, hydrophones that convert changes in water pressure to electrical signals, ultrasonic transducers that transmit and receive ultrasound waves, and so on. When the sensors are located sufficiently close together, they behave as a collective entity, called the array. The sensors in passive arrays are receivers of the waves generated by unknown remote sources. Active arrays have sensors that play the dual role of sources and receivers. The sources emit waves that propagate through the medium and are scattered back to the array, where they are captured by the receivers.

The recordings at the receivers are called the array data. The coherent imaging process seeks to transform the data into an imaging function that peaks in the support of the unknown sources or scatterers. The key data processing step in the image formation is the synchronization of the received signals using a mathematical model of wave back-propagation from the array to a search (imaging) location $\vec{\mathbf{r}}_s$. The expectation is that when $\vec{\mathbf{r}}_s$ lies in the support of the unknown sources or scatterers, the recordings are synchronized and add up over the sensors to give a peak value of the imaging function. Indeed, this happens if we have an accurate model of wave propagation between the array and the imaging region.

Image formation is somewhat similar to the time reversal process. Time reversal is an experiment that uses an active array to first receive the waves from a remote source, and then re-emit the time-reversed recordings back in the medium. The wave equation is time reversible if the medium is non-dissipative, so the waves are expected to propagate back to the source and focus near it. The focusing resolution in the range direction is mostly affected by the bandwidth of the emitted signals. The focusing in cross-range (i.e., in the plane orthogonal to the range) depends on many factors, such as how large the array is, how far the source is, and how much scattering occurs between the source and receiver. The focusing is never perfect, because the array cannot capture the whole wave field emitted from the source. Thus, the time reversal process is not exactly like solving the wave equation backward in time to recover the initial source. The expectation is that the larger the array is, the better the focus, because more of the waves are captured and turned back. But no matter how large the array is, even if it surrounds the source, the evanescent waves cannot reach it, and the focusing resolution is limited by diffraction. For example, the resolution of refocusing of time harmonic waves cannot exceed the Abbe diffraction limit of $\lambda/2$, where λ is the wavelength **[12]**. In most applications the array is supported in a set of small diameter (aperture) $|\mathcal{A}|$ with respect to the distance L of propagation, and the expected resolution of refocusing in cross-range is given by the Rayleigh formula $\lambda L/|\mathcal{A}|$ if there is no scattering in the medium. The interesting property of time reversal, known as *super-resolution*, is that scattering may improve the cross-range focusing. This is demonstrated and explained in **[15, 16, 26]**, and it is analyzed theoretically and numerically in **[5, 4, 30]**. The latter studies are based on random models of the scattering medium, and introduce the important concept of *statistical stability.* Stability means that the focusing is robust, it is independent of the realization of the random medium, it is observable and reproducible. Stability is guaranteed in general only for sources emitting broad-band signals, as pointed out in **[5]**, although there are regimes where it holds for narrowband signals as long as the sources and arrays have sufficiently large support **[30]**.

The super-resolution and robustness of time reversal are due entirely to the waves propagating back in exactly the same medium they came from. The observer has access to the vicinity of the source and can see the focus there, whereas in imaging the access is limited to the array. Moreover, the medium is not known in detail in many imaging applications. For example, in underwater acoustics, the smooth part of the wave speed is known, but there are small scale fluctuations due to internal waves **[17]** that cannot be known, and are not even of interest in imaging. It is the uncertainty about such small scale inhomogeneities that we model with random spatial processes, and thus speak of imaging in random media.

The array data processing for image formation may appear to mimic the back-propagation of the waves in the time reversal process, but there is a fundamental difference. While in time reversal the waves go back in the same medium, the back-propagation in imaging is done mathematically, using a fictitious medium model that incorporates the large scale features of the wave speed, but not its fluctuations, which are unknown. This difference is essential in regimes with strong cumulative scattering of the waves by the random inhomogeneities. The longer the waves travel in the random medium, the stronger the scattering, which leads to loss of coherence of the wave field. The coherent waves are useful in imaging because we can relate the locations of the sought-after sources or scatterers to their arrival time. Multiple scattering by the inhomogeneities transfers energy to the incoherent part of the waves, the random fluctuations which are unwanted reverberations. We may think of the reverberations as *noise*, but we should remember that they are not ordinary additive, uncorrelated noise. They have complicated statistical structure, with correlations across the array and in the bandwidth, and are difficult to mitigate. Our goal is to quantify the deterioration of resolution and robustness of images in random media, and to propose efficient methods for mitigating cumulative scattering effects.

The data processing in image formation must take into account the presence of scattering boundaries, because they create multiple traveling paths of the waves received at the array. This happens in waveguides, where boundaries trap the waves and guide the energy in a preferred direction, the waveguide axis, along which the medium is unbounded. To analyze the wave field in waveguides, we may decompose it mathematically in an infinite, countable set of monochromatic waves called *waveguide modes*. In ideal waveguides, with flat boundaries and wave-speed that is constant or varies smoothly in the waveguide cross-section, the modes are uncoupled. Finitely many of them propagate along the waveguide axis at different speeds, and we may associate them to plane waves with different angles of incidence at the boundary. The slower modes correspond to nearly normal incidence. They bounce off the boundaries many times, thus taking a long path from the source to the array. The faster modes correspond to small grazing angles at the boundary, and shorter paths to the array. The remaining infinitely many modes are evanescent, with amplitudes decaying exponentially with the distance along the waveguide axis.

We are concerned with waveguides that are randomly perturbed versions of the ideal ones. They may have boundaries with small random fluctuations, and/or internal inhomogeneities that cause small fluctuations of the wave speed. We present a rigorous asymptotic theory of wave propagation in such waveguides, where the asymptotics is in the small parameter that measures the magnitude of the fluctuations. We show that when the waves travel sufficiently far from the source, the small fluctuations cause significant cumulative scattering. It amounts to coupling of the waveguide modes, and gradual loss of coherence of their amplitudes. We give a detailed analysis of mode coupling and quantify the net scattering via three important length scales: the *scattering mean free path*, the *transport mean free path* and the *equipartition distance*. The mode dependent scattering mean free path is the distance over which the mode loses its coherence, meaning that its random fluctuations dominate its statistical mean. The transport mean free path is classically defined as the distance beyond which the waves forget their initial direction [**31**]. Because the modes are associated to directions of incidence at the boundary, we define the mode dependent transport mean free path as the length scale over which the mode exchanges energy with the other modes. The equipartition distance is the longest of the three length scales, and it gives the distance over which scattering distributes the energy uniformly among the modes, independent of the initial source excitation.