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DIMERS AND ORTHOGONAL POLYNOMIALS: CONNECTIONS WITH RANDOM MATRICES

by

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Abstract. – In these lecture notes we present some connections between random matrices, the asymmetric exclusion process, random tilings. First we consider the Gaussian Unitary Ensemble of random matrices. We analyze the mathematical structure of their eigenvalues as well as the eigenvalues of the principal minors. Then we introduce a one-dimensional interacting particle system (the TASEP) where a similar mathematical structure arises. Finally we extend the particle system to a 2+1 dimensional model. This model has a marginal given by the TASEP and the fixed time projection is a random tiling measure. A special case in discrete time gives the well-known Aztec diamond.

Résumé. – Dans ces notes, nous présentons quelques connexions entre les matrices aléatoires, le processus d'exclusion simple et les pavages aléatoires. D'abord, nous considérons l'ensemble gaussien unitaire des matrices aléatoires. Nous analysons la structure mathématique de leurs valeurs propres, ainsi que celles des mineurs principaux. Nous introduisons ensuite un système de particules en interaction en dimension 1 (le TASEP) dans lequel une structure mathématique similaire est exhibée. Enfin, nous étendons le système de particules à un modèle en dimension $2 + 1$. Ce modèle a une marginale donnée par le TASEP et sa projection à un instant fixé est une mesure sur des pavages aléatoires. Un cas spécial en temps discret donne le modèle bien connu du diamant aztèque.

1. Structure of these lecture notes

In these notes we explain why there are limit processes and distribution functions which arise in random matrix theory, interacting particle systems, stochastic growth models, and random tilings models. This is due to a common mathematical structure describing special models in the different fields. In Section 2 we introduce the mathematical structure in the context of the Gaussian Unitary Ensemble and its

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eigenvalues' minor process. In Section 3 we introduce the totally asymmetric simple exclusion process (TASEP), a particle process sharing the same structure as the GUE minor process. Finally, in Section 4 we discuss the extension of TASEP to an interacting particles system in $2+1$ dimensions. This model has two projections which are still Markov processes [**11**] (see also the lecture notes [**30**]):

1. TASEP,
2. the Charlier process (a discrete space analogue of Dyson's Brownian motion).

Furthermore, projections at fixed times of the model leads to random tilings measures, one of which is the measure arising from the well known shuffling algorithm for the Aztec diamond.

Some books in random matrix theory are [**48, 3, 34**]. In the handbook [**1**] one finds a lot of applications of random matrices and related models. For instance, the relation between random matrices and growth models is discussed in [**33**], while determinantal point processes are explained in [**8**].

2. Gaussian Unitary Ensemble of random matrices (GUE)

2.1. The Gaussian Ensembles of random matrices. – The Gaussian ensembles of random matrices have been introduced by physicists (Dyson, Wigner, ...) in the sixties to model statistical properties of the resonance spectrum of heavy nuclei. The energy levels of a quantum system are the eigenvalues of a Hamiltonian. They observed that statistical properties such as eigenvalues' spacing statistics is the roughly the same for all heavy nuclei, i.e., there is a universal behavior. Based on these observations, they had the brilliant idea to study the statistical properties by considering a *random Hamiltonian*. Further, since the heavy nuclei have a lot of bound states, their Hamiltonian was replaced by a *large matrix with random entries*. Finally, to have a chance to describe the physical properties of heavy atoms, the matrices need to satisfy the intrinsic symmetries of the systems:

1. a *real symmetric* matrix can describe a system with *time reversal symmetry* and *rotation invariance or integer magnetic momentum*,
2. a *real quaternionic* matrix (i.e., the basis are the Pauli matrices) can be used for *time reversal symmetry* and *half-integer magnetic momentum*,
3. a *complex hermitian* matrix can describe a system which is *not time reversal invariant* (e.g., with external magnetic field).

This lead to the definition of the Gaussian Ensembles of random matrices. In this lecture notes we consider only the case of complex hermitian matrices.

Definition 1. – *The* Gaussian Unitary Ensemble *(GUE) of random matrices is a probability measure* $\mathbb{P}$ *on the set of* $N \times N$ *complex hermitian matrices given by*

$$\mathbb{P}(\mathrm{d}H) = \frac{1}{Z_N} \exp\Big(-\frac{\beta}{4N} \operatorname{Tr}(H^2)\Big)\, \mathrm{d}H, \quad \text{with } \beta = 2, \tag{1}$$

TABLE 1. Typical scaling for the Gaussian Unitary Ensemble

	$a = 1/2N$	$a = 1$	$a = N$
Largest eigenvalue	$2N + \mathcal{O}(N^{1/3})$	$\sqrt{2N} + \mathcal{O}(N^{-1/6})$	$\sqrt{2} + \mathcal{O}(N^{-2/3})$
Eigenvalues density	$\mathcal{O}(1)$	$\mathcal{O}(N^{1/2})$	$\mathcal{O}(N)$

where $\mathrm{d}H = \prod_{i=1}^{N} \mathrm{d}H_{i,i} \prod_{1 \le i < j \le N} \mathrm{dRe}(H_{i,j}) \mathrm{dIm}(H_{i,j})$ *is the reference measure, and* Z_N *is the normalization constant.*

The meaning of $\beta = 2$ will be clear once we consider the induced measure on the eigenvalues. The name GUE refers to the Gaussian form of the measure (1) and its invariance over the unitary transformations. From a physical point of view, this invariance holds for systems which do not depend on the choice of basis used to describe them. By imposing that the measure $\mathbb{P}$ is (a) invariant under the change of basis (in the present case, invariant under the action of the group of symmetry $U(N)$) and (b) the entries of the matrices are independent random variables (of course, up to the required symmetry), then the only solutions are measures of the form

$$\exp\left(-a \operatorname{Tr}(H^2) + b \operatorname{Tr}(H) + c\right), \quad a > 0, b, c \in \mathbb{R}. \tag{2}$$

The value of c is determined by the normalization requirement, while by an appropriate shift of the zero of the energy (i.e., $H \to H - E$ for some given E), we can set $b = 0$. The energy shift is irrelevant from the physical point of view because by the first principle of thermodynamics, the energy of a system is an extensive observable defined *up to a constant*. The value of a is a scale parameter that can be freely chosen. In the literature there are mainly three typical choices, see Table 1.

Another way to obtain (1) is to take the random variables, $H_{i,i} \sim \mathcal{N}(0, N)$ for $i = 1, \dots, N$, and $\mathrm{Re}(H_{i,j}) \sim \mathcal{N}(0, N/2)$, $\mathrm{Im}(H_{i,j}) \sim \mathcal{N}(0, N/2)$ for $1 \le i < j \le N$ to be independent random variables.

For the real symmetric (resp. quaternionic) class of matrices, one defines the Gaussian Orthogonal Ensemble (GOE) (resp. Gaussian Symplectic Ensemble (GSE)) as in Definition 1 but with $\beta = 1$ (resp. $\beta = 4$) and, of course, the reference measure is now the product Lebesgue measure over the independent entries of the matrices.

2.2. Eigenvalues' distribution. – One quantity of interest for random matrices is the distribution of the eigenvalues. The invariance under the choice of basis for the Gaussian ensembles of random matrices implies that the distribution of the eigenvalues can be explicitly computed with the following result. Denote by $P_{\mathrm{GUE}}(\lambda)$ the probability density of eigenvalues at $\lambda \in \mathbb{R}^N$.

Proposition 2. – *Let $\lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{R}$ denote the N eigenvalues of a random matrix H with law* (1). *Then, the joint density of the eigenvalues is given by*

$$P_{\mathrm{GUE}}(\lambda) = \frac{1}{Z_N} |\Delta_N(\lambda)|^\beta \prod_{i=1}^N \exp\left(-\frac{\beta}{4N}\lambda_i^2\right), \quad \text{with } \beta = 2, \tag{3}$$

$\Delta_N(\lambda) := \prod_{1\le i<j\le N}(\lambda_j - \lambda_i)$ is the Vandermonde determinant, and Z_N is a normalization constant.

The Vandermonde determinant, Δ_N, is called a determinant because of the identity

$$\Delta_N(\lambda) = \det\left[\lambda_i^{j-1}\right]_{1\le i,j\le N}. \tag{4}$$

Notice that $P_{\mathrm{GUE}}(\#\text{ e.v. } \in [x, x+\mathrm{d}x]) \sim (\mathrm{d}x)^2$, so that the probability of having eigenvalues with multiplicity greather of equal to two is zero. In this case, the *point process* of the eigenvalues, $\sum_{n=1}^N \delta_{\lambda_n}$, is called *simple*.

For GOE (resp. GSE) the joint distributions of eigenvalues have the form (3) but with $\beta = 1$ (resp. $\beta = 4$) instead.

2.3. Orthogonal polynomials. – The correlation function for GUE eigenvalues can be described using Hermite orthogonal polynomials. Therefore, we briefly discuss orthogonal polynomials on $\mathbb{R}$. Formulas can easily be adapted for polynomials on $\mathbb{Z}$ by replacing the Lebesgue measure by the counting measure and integrals by sums.

Definition 3. – *Given a weight $\omega : \mathbb{R} \mapsto \mathbb{R}_+$, the* orthogonal polynomials $\{q_k(x), k \ge 0\}$ *are defined by the following two conditions:*

1. *$q_k(x)$ is a polynomial of degree k with $q_k(x) = u_k x^k + \dots$, $u_k > 0$,*
2. *the $q_k(x)$ satisfy the orthonormality condition,*

$$\langle q_k, q_l\rangle_\omega := \int_{\mathbb{R}} \omega(x) q_k(x) q_l(x) \mathrm{d}x = \delta_{k,l}. \tag{5}$$

Remark 4. – There are other normalizations which are often used, such as in the *Askey Scheme of hypergeometric orthogonal polynomials* [**47**]. Sometimes, the polynomials are taken to be *monic*, i.e., $u_k = 1$ and the orthonormality condition is replaced by an orthogonality condition $\int_{\mathbb{R}} \omega(x)\tilde{q}_k(x)\tilde{q}_l(x)\mathrm{d}x = c_k\delta_{k,l}$. Of course $\tilde{q}_k(x) = q_k(x)/u_k$ and $c_k = 1/u_k^2$. Sometimes, the polynomials are neither orthonormal (like in Definition 3) nor monic, like the standard Hermite polynomials that we will encounter, and are given by derivatives of a generating function.

A useful formula for sums of orthogonal polynomials is the *Christoffel-Darboux formula*:

$$\sum_{k=0}^{N-1} q_k(x)q_k(y) = \begin{cases} \dfrac{u_{N-1}}{u_N}\dfrac{q_N(x)q_{N-1}(y) - q_{N-1}(x)q_N(y)}{x-y}, & \text{for } x \ne y, \\ \dfrac{u_{N-1}}{u_N}(q_N'(x)q_{N-1}(x) - q_{N-1}'(x)q_N(x)), & \text{for } x = y. \end{cases} \tag{6}$$