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Abstract. – We explain Tits' structure theory for smooth connected unipotent groups over general fields of positive characteristic (especially imperfect fields). This builds on earlier work of Rosenlicht [**9**] and concerns the structure of smooth connected unipotent groups as well as torus actions on such groups over an arbitrary ground field of positive characteristic. We use it to establish a general structure theorem for solvable smooth connected affine k-groups that replaces (and generalizes) the semi-direct product structure over perfect k.

***Résumé* (La structure des groupes résolubles sur des corps généraux).** – Nous expliquons la théorie de structure de Tits des groupes algébriques unipotents connexes et lisses sur un corps général de caractéristique positive (en particulier imparfait). Ceci s'appuie sur les travaux antérieurs de Rosenlicht [**9**] concernant la structure des groupes unipotents lisses et connexes ainsi que des actions de tores sur ces groupes au-dessus d'un corps de base de caractéristique positive. Nous l'utilisons pour établir un théorème de structure plus général pour les k--groupes affines résolubles lisses et connexes qui remplace (et généralise) la structure de produit semi-direct dans le cas d'un corps parfait k.

Introduction

Consider a smooth connected solvable affine group G over a field k. If k is algebraically closed then $G = T \ltimes \mathscr{R}_u(G)$ for any maximal torus T of G [**1**, 10.6(4)]. Over more general k, an analogous such semi-direct product structure can fail to exist.

For example, consider an imperfect field k of characteristic $p > 0$ and $a \in k - k^p$, so $k' := k(a^{1/p})$ is a degree-p purely inseparable extension of k. Note that $k'_s := k' \otimes_k k_s = k_s(a^{1/p})$ is a separable closure of k', and ${k'_s}^p \subset k_s$. The affine Weil restriction $G = \mathrm{R}_{k'/k}(\mathbf{G}_\mathrm{m})$ is an open subscheme of $\mathrm{R}_{k'/k}(\mathbf{A}^1_{k'}) = \mathbf{A}^p_k$, so it is a smooth connected affine k-group of dimension $p > 1$. Loosely speaking, G is "${k'}^\times$ viewed as a k-group". More precisely, for k-algebras R we have $G(R) = (k' \otimes_k R)^\times$ functorially in R. (See Exercise U.4 for a treatment of Weil restriction in the affine case.) The commutative k-group G contains an evident 1-dimensional torus $T \simeq \mathbf{G}_\mathrm{m}$

corresponding to the subgroup $R^\times \subset (k' \otimes_k R)^\times$, and G/T is unipotent because $(G/T)(k_s) = (k'_s)^\times/(k_s)^\times$ is p-torsion. In particular, T is the unique maximal torus of G. Since the group $G(k_s) = {k'_s}^\times$ has no nontrivial p-torsion, G contains *no* nontrivial unipotent smooth connected k-subgroup. Thus, G is a commutative counterexample over k to the analogue of the semi-direct product structure for connected solvable smooth affine groups over $\overline{k}$.

The appearance of imperfect fields in the preceding counterexample is essential. To explain this, recall Grothendieck's theorem that over a general field k, if S is a maximal k-torus in a smooth affine k-group H then $S_{\overline{k}}$ is maximal in $H_{\overline{k}}$. (This theorem is an application of [**5**, XIV, Thm. 1.1] to the smooth affine k-group $Z_H(S)$, since a "maximal torus" over k in the sense of [**5**, XII, Def. 1.3] is *defined* to be a k-torus that is maximal after scalar extension to $\overline{k}$. For another proof, see [**3**, A.1.2].) Thus, by the conjugacy of maximal tori in $G_{\overline{k}}$, $G = T \ltimes U$ for a k-torus T and a unipotent smooth connected normal k-subgroup $U \subset G$ if and only if the subgroup $\mathscr{R}_u(G_{\overline{k}}) \subset G_{\overline{k}}$ is defined over k (i.e., descends to a k-subgroup of G). In such cases, the semi-direct product structure holds for G over k using any maximal k-torus T of G (and U is unique: it must be a k-descent of $\mathscr{R}_u(G_{\overline{k}})$). If k is perfect then by Galois descent we may always descend $\mathscr{R}_u(G_{\overline{k}})$ to a k-subgroup of G. The main challenge is the case of imperfect k.

The results we shall discuss for unipotent groups were presented by Tits in a course at Yale University in 1967, and lecture notes [**12**] for that course were circulated but never published. Much of the course was concerned with general results on linear algebraic groups that are available now in many standard references (such as [**1**], [**7**], and [**11**]). The original account (with proofs) of Tits' structure theory of unipotent groups is his unpublished Yale lecture notes, and a summary of the results is given in [**8**, Ch. V].

Our exposition in §1–§4 is an improvement of [**4**, App. B] via simplifications in some proofs. In some parts we have simply reproduced arguments from Tits' lecture notes. The general solvable case is addressed in §5, where we include applications to general smooth connected affine k-groups. *Throughout the discussion below, k is an arbitrary field with characteristic $p > 0$.*

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1. Subgroups of vector groups

The *additive group* is denoted $\mathbf{G}_{\mathrm{a}}$ and the *multiplicative group* is denoted $\mathbf{G}_{\mathrm{m}}$, always with the base ring understood from context.

Definition 1.1. – A *vector group* over a field k is a smooth commutative k-group V that admits an isomorphism to $\mathbf{G}_{\mathrm{a}}^n$ for some $n \geqslant 0$. The $\mathbf{G}_{\mathrm{m}}$-scaling action arising from such an isomorphism is a *linear structure* on V.

Observe that the $\mathbf{G}_{\mathrm{m}}$-action on V arising from a linear structure induces the canonical $k^{\times}$-action on $\mathrm{Lie}(V)$ (e.g., if $\mathrm{char}(k) = p > 0$ then the composition of such a $\mathbf{G}_{\mathrm{m}}$-action on V with the p-power map on $\mathbf{G}_{\mathrm{m}}$ does not arise from a linear structure on V when $V \neq 0$).

Example 1.2. – If W is a finite-dimensional k-vector space then the *associated vector group* $\underline{W}$ represents the functor $R \rightsquigarrow R \otimes_k W$ on k-algebras and its formation commutes with any extension of the ground field. Explicitly, $\underline{W} = \mathrm{Spec}(\mathrm{Sym}(W^*))$ and it has a unique linear structure relative to which the natural identification of groups $\underline{W}(k_s) \simeq W_{k_s}$ carries the linear structure over to the $k_s^{\times}$-action on W_{k_s} arising from the k_s-vector space structure; call this the *canonical* linear structure on $\underline{W}$. (We can use k instead of k_s in this characterization when k is infinite, as $W(k)$ is Zariski-dense in $\underline{W}$ for infinite k.) For finite-dimensional k-vector spaces W and W', the subset $\mathrm{Hom}_k(W, W') \subset \mathrm{Hom}_{k\text{-gp}}(\underline{W}, \underline{W'})$ consists of precisely the k-homomorphisms respecting the canonical linear structures.

When linear structures are specified on a pair of vector groups, a homomorphism respecting them is called *linear*. Over a field of characteristic 0 there is a unique linear structure and all homomorphisms are linear. Over a field with characteristic $p > 0$ the linear structure is not unique in dimension larger than 1 (e.g., $a.(x,y) := (ax + (a - a^p)y^p, ay)$ is a linear structure on $\mathbf{G}_{\mathrm{a}}^2$, obtained from the usual one via the non-linear k-group automorphism $(x, y) \mapsto (x + y^p, y)$ of $\mathbf{G}_{\mathrm{a}}^2$). For a finite-dimensional k-vector space W, a *linear subgroup* of $\underline{W}$ is a smooth closed k-subgroup that is stable under the $\mathbf{G}_{\mathrm{m}}$-action. By computing with k_s-points and using Galois descent, it is straightforward to verify that the linear subgroups of $\underline{W}$ are precisely $\underline{W'}$ for k-subspaces $W' \subset W$.

Definition 1.3. – A smooth connected solvable k-group G is *k-split* if it admits a composition series

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = 1$$

consisting of smooth closed k-subgroups such that G_{i+1} is normal in G_i and the quotient G_i/G_{i+1} is k-isomorphic to $\mathbf{G}_{\mathrm{a}}$ or $\mathbf{G}_{\mathrm{m}}$ for all $0 \leqslant i < n$. (Such G_i must be connected, so each G_i is also a k-split smooth connected solvable k-group.)

In the case of tori this is a widely-used notion, and it satisfies convenient properties, such as: (i) every subtorus or quotient torus (over k) of a k-split k-torus is k-split, (ii) every k-torus is an almost direct product of its maximal k-split subtorus and its maximal k-anisotropic subtorus. However, in contrast with the case of tori, it is not true for general smooth connected solvable G that the k-split property is inherited by smooth connected normal k-subgroups:

***Example 1.4* (Rosenlicht).** – Assume k is imperfect and choose $a \in k - k^p$. The k-group

$$\mathbf{U} := \{y^p = x - ax^p\}$$

is a k-subgroup of the k-split $G = \mathbf{G}_{\mathrm{a}}^2$ and it becomes isomorphic to $\mathbf{G}_{\mathrm{a}}$ over $k(a^{1/p})$ but there is no non-constant k-morphism $f : \mathbf{A}_k^1 \to \mathbf{U}$, let alone a k-group isomorphism $\mathbf{G}_{\mathrm{a}} \simeq \mathbf{U}$. Indeed, the regular compactification $\overline{\mathbf{U}}$ of $\mathbf{U}$ has a unique point $\infty_{\mathbf{U}} \in \overline{\mathbf{U}} - \mathbf{U}$, and the regular compactification of $\mathbf{G}_{\mathrm{a}}$ is $\mathbf{P}_k^1$ via $x \mapsto [x, 1]$, so any non-constant map f extends to a (finite) surjective map $\mathbf{P}_k^1 \to \overline{\mathbf{U}}$ that must carry $[1, 0]$ to $\infty_{\mathbf{U}}$, an absurdity since $k(\infty_{\mathbf{U}}) = k(a^{1/p}) \neq k$.

Tits introduced an analogue for unipotent k-groups of the notion of anisotropicity for tori over a field. This rests on a preliminary understanding of the properties of subgroups of vector groups, so we take up that study now. The main case of interest to us will be imperfect ground fields, due to the fact that every unipotent smooth connected group over a perfect field is split (see Exercises U.9(iii)).

***Definition 1.5*.** – A polynomial $f \in k[x_1, \dots, x_n]$ is a *p-polynomial* if every monomial appearing in f has the form $c_{ij} x_i^{p^j}$ for some $c_{ij} \in k$; that is, $f = \sum f_i(x_i)$ with $f_i(x_i) = \sum_j c_{ij} x_i^{p^j} \in k[x_i]$. (In particular, $f_i(0) = 0$ for all i. Together with the identity $f = \sum f_i(x_i)$, this uniquely determines each f_i in terms of f. Note that $f(0) = 0$.)

***Proposition 1.6*.** – *A polynomial $f \in k[x_1, \dots, x_n]$ is a p-polynomial if and only if the associated map of k-schemes $\mathbf{G}_{\mathrm{a}}^n \to \mathbf{G}_{\mathrm{a}}$ is a k-homomorphism.*

Proof. – This is elementary and is left to the reader. □

A nonzero polynomial over k is *separable* if its zero scheme in affine space is generically k-smooth.

***Proposition 1.7*.** – *Let $f \in k[x_1, \dots, x_n]$ be a nonzero polynomial such that $f(0) = 0$. Then the subscheme $f^{-1}(0) \subset \mathbf{G}_{\mathrm{a}}^n$ is a smooth k-subgroup if and only if f is a separable p-polynomial.*

Proof. – The "if" direction is clear. For the converse, we assume that $f^{-1}(0)$ is a smooth k-subgroup and we denote it as G. The smoothness implies that f is separable. To prove that f is a p-polynomial, by Proposition 1.6 it suffices to prove that the associated map of k-schemes $\mathbf{G}_{\mathrm{a}}^n \to \mathbf{G}_{\mathrm{a}}$ is a k-homomorphism. Without loss of generality, we may assume that k is algebraically closed.

For any $\alpha \in G(k)$, $f(x + \alpha)$ and $f(x)$ have the same zero scheme (namely, G) inside $\mathbf{G}_{\mathrm{a}}^n$. Thus, $f(x + \alpha) = c(\alpha) f(x)$ for a unique $c(\alpha) \in k^\times$. Consideration of a highest-degree monomial term appearing in f implies that $c = 1$. Pick $\beta \in k^n$, so $f(\beta + \alpha) - f(\beta) = 0$ for all $\alpha \in G(k)$. Thus $f(\beta + x) - f(\beta)$ vanishes on G, so $f(\beta + x) - f(\beta) = g(\beta) f(x)$ for a unique $g(\beta) \in k$. Consideration of a highest-degree monomial term in f forces $g(\beta) = 1$. □