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by

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Abstract. – We study the following phenomenon: some *non-split* connected semisimple $\mathbf{Q}$-groups G admit flat affine $\mathbf{Z}$-group models $\mathscr{G}$ with "everywhere good reduction" (i.e., $\mathscr{G}_{\mathbf{F}_p}$ is a connected semisimple $\mathbf{F}_p$-group for every prime p). Moreover, considering such $\mathscr{G}$ up to $\mathbf{Z}$-group isomorphism, there can be more than one such $\mathscr{G}$ for a given G. This is seen classically for types B and D by using positive-definite quadratic lattices.

The study of such $\mathbf{Z}$-groups provides concrete applications of many facets of the theory of reductive groups over rings (scheme of Borel subgroups, automorphism scheme, relative non-abelian cohomology, etc.), and it highlights the role of number theory (class field theory, mass formulas, strong approximation, point-counting over finite fields, etc.) in analyzing the possibilities. In part, this is an expository account of [**26**].

***Résumé* (Groups réductifs non-déployés sur Z).** – Nous étudions le phénomène suivant: certains $\mathbf{Q}$-groupes G semi-simples connexes *non déployés* admettent comme modèles des $\mathbf{Z}$-groupes $\mathscr{G}$ affines et plats avec "partout bonne réduction" (c'est à dire, $\mathscr{G}_{\mathbf{F}_p}$ est un $\mathbf{F}_p$-groupe semi-simple pour chaque premier p). En outre, considérant de tels $\mathscr{G}$ à $\mathbf{Z}$-groupe isomorphisme près, il y a au plus un tel $\mathscr{G}$ pour un G donné. Ceci est vu classiquement pour les types B et D en utilisant des réseaux quadratiques définis positifs.

L'étude de ces $\mathbf{Z}$-groupes donne lieu à des applications concrètes d'aspects multiples, de la théorie des groupes réductifs sur des anneaux (schémas de sous-groupes de Borel, schémas d'automorphismes, cohomologie relative non abélienne, etc.), et met en évidence le rôle de la théorie des nombres (théorie du corps de classes, formules de masse, approximation forte, comptage de points sur les corps finis, etc.) dans l'analyse des possibilités. En partie, ceci est un article d'exposition sur [**26**].

1. Chevalley groups and Z-models

A *Chevalley group* is a reductive $\mathbf{Z}$-group scheme (i.e., a smooth affine group scheme $G \to \mathrm{Spec}(\mathbf{Z})$ with connected reductive fibers) that admits a fiberwise maximal $\mathbf{Z}$-torus $T \subset G$. For example, the classical groups SL_n, GL_n, PGL_n, Sp_{2n}, and SO_n over $\mathbf{Z}$ are all Chevalley groups. (The characteristic-free definition of SO_n requires some care when n is even; see [**14**, C.2.10].) Many authors require Chevalley groups to have semisimple fibers, but this is a matter of convention.

A more traditional viewpoint on Chevalley groups is obtained via the notion of $\mathbf{Z}$-*model* of a connected reductive $\mathbf{Q}$-group. In general, if K is the fraction field of a domain R then an R-*model* of a connected reductive K-group G is a pair $(\mathscr{G}, \theta)$ consisting of a reductive R-group scheme $\mathscr{G}$ and an isomorphism of K-groups $\theta : \mathscr{G}_K \simeq G$. The notion of isomorphism between models of G is defined in an evident manner. (Our notion of "model" is more restrictive than in other circumstances, where one allows any flat and finitely presented – or perhaps even smooth – affine group with a specified generic fiber.)

Lemma 1.1. – *The generic fiber of any Chevalley group is split.*

Proof. – It suffices to show that any $\mathbf{Z}$-torus is necessarily split. By [**20**, X, 1.2, 5.16] (or [**14**, Cor. B.3.6]), the category of tori over a connected normal noetherian scheme S is anti-equivalent to the category of finite free $\mathbf{Z}$-modules equipped with a continuous action of $\pi_1(S)$. (When $S = \mathrm{Spec}(k)$ for a field k, this recovers the familiar "character lattice" construction for k-tori.) An S-torus is split when the associated $\pi_1(S)$-action is trivial.

For any Dedekind domain A, the connected finite étale covers of $\mathrm{Spec}(A)$ correspond to the finite extensions of Dedekind domains $A \hookrightarrow A'$ with unit discriminant. Thus, by Minkowski's theorem that every number field $K \neq \mathbf{Q}$ has a ramified prime, $\mathrm{Spec}(\mathbf{Z})$ has no nontrivial connected finite étale covers. Hence, $\pi_1(\mathrm{Spec}(\mathbf{Z})) = 1$, so all $\mathbf{Z}$-tori are split. $\square$

Every Chevalley group $\mathscr{G}$ is a $\mathbf{Z}$-model of its split connected reductive generic fiber over $\mathbf{Q}$, and the Existence and Isomorphisms Theorems over $\mathbf{Z}$ provide a converse that is one of the main theorems of [**20**]:

***Theorem 1.2* (Chevalley, Demazure)**. – *Let R be a domain with fraction field K. Every split connected reductive K-group G admits an R-model of the form $\mathbf{G}_R$ for a Chevalley group $\mathbf{G}$ over $\mathbf{Z}$, and $\mathbf{G}$ is uniquely determined up to $\mathbf{Z}$-group isomorphism.*

The existence of $\mathbf{G}$ for each G was first proved for $K = \mathbf{Q}$ as the main result in [**10**], though the language of reductive group schemes over $\mathbf{Z}$ was not available at that time. The approach used by Demazure in [**20**, XXV] is to abstractly build a "split" $\mathbf{Z}$-group $\mathbf{G}$ whose associated root datum may be specified in advance. The Isomorphism Theorem for split connected reductive groups over K then ensures that one gets all such K-groups as generic fibers of the $\mathbf{G}_R$'s by varying over all possibilities for the root datum. Chevalley groups are the *only* $\mathbf{Z}$-models in the split case over $\mathbf{Q}$, so we get a characterization of Chevalley groups without any mention of maximal tori over rings. More generally:

Proposition 1.3. – *If R is a principal ideal domain and G is a split connected reductive group over $K = \mathrm{Frac}(R)$ then any R-model of G is $\mathbf{G}_R$ for a Chevalley group $\mathbf{G}$ over $\mathbf{Z}$.*

The hypothesis on R is optimal: if R is Dedekind with fraction field K and I is a nonzero ideal in R whose class in $\mathrm{Pic}(R)$ is not a square then $\mathrm{SL}(R \oplus I)$ is a *non-trivial* Zariski-form of $\mathrm{SL}_{2,R}$ (see [**14**, Exer. 7.3.10]). We postpone the proof of Proposition 1.3 until §3, as it requires cohomological notions introduced there.

The preceding discussion is summarized by:

Theorem 1.4. – *Passage to the* $\mathbf{Q}$*-fiber defines a bijection from the set of* $\mathbf{Z}$*-isomorphism classes of Chevalley groups onto the set of isomorphism classes of split connected reductive* $\mathbf{Q}$*-groups, with each set classified by root data* (*up to isomorphism*). *Moreover, the* only $\mathbf{Z}$*-models of such* $\mathbf{Q}$*-groups are those provided by Chevalley groups.*

Work of Chevalley ([**12**], [**10**]) and Demazure–Grothendieck [**20**] provides a satisfactory understanding of this remarkable theorem. (For any scheme $S \neq \varnothing$, [**20**, XXII, 1.13] provides a definition of *Chevalley* S*-group* avoiding the crutch of the theory over $\mathbf{Z}$. This involves additional conditions that are automatic for $S = \mathrm{Spec}(\mathbf{Z})$.)

Informally, the connected semisimple $\mathbf{Q}$-groups arising as generic fibers of non-Chevalley semisimple $\mathbf{Z}$-groups are those with "good reduction" at all primes but non-split over $\mathbf{R}$ (see Propositions 3.12 and 4.10). The theory surrounding such $\mathbf{Z}$-groups was the topic of [**26**], where the possibilities for the $\mathbf{Q}$-fiber were classified (under an absolutely simple hypothesis) and some explicit $\mathbf{Z}$-models were given for exceptional types, generalizing examples arising from quadratic lattices.

Overview. In §2 we discuss special orthogonal groups in the scheme-theoretic framework, highlighting the base scheme $\mathrm{Spec}(\mathbf{Z})$ and some classical examples of semisimple $\mathbf{Z}$-groups with non-split generic fiber arising from quadratic lattices. In §3 we discuss general cohomological formalism for working with smooth (or more generally, fppf) affine groups over rings, extending the more widely-known formalism over fields as in [**45**, III].

In §4 we describe the possibilities for the generic fibers of reductive $\mathbf{Z}$-groups, with an emphasis on the case of semisimple $\mathbf{Z}$-groups whose fibers are absolutely simple and simply connected, and we show that this case accounts for the rest via direct products and central isogenies. In §5 we introduce Coxeter's order in Cayley's definite octonion algebra over $\mathbf{Q}$, and we use it in §6 to describe some non-split examples over $\mathbf{Z}$. In §7 we explain (following [**26**]) how to use mass formulas to prove in some cases that the list of $\mathbf{Z}$-models found in §6 for certain $\mathbf{Q}$-groups is exhaustive.

In Appendix A we use the cohomological formalism of semisimple $\mathbf{Z}$-groups to prove that an indefinite non-degenerate quadratic lattice over $\mathbf{Z}$ is determined up to isomorphism by its signature (in odd rank these are not unimodular lattices), and in Appendix B we discuss generalities concerning octonion algebras over commutative rings, with an emphasis on the special case of Dedekind domains. Finally, in Appendix C we discuss an explicit construction of the simply connected Chevalley group of type E_6.

Justification of the construction of simply connected Chevalley groups over $\mathbf{Z}$ of types F_4 and E_6 via Jordan algebras (in §6 and Appendix C) uses concrete linear algebra and Lie algebra computations over $\mathbf{Z}$ via Mathematica code written by Jiu-Kang Yu (see [**51**]); for E_6 this is only needed with local problems at $p = 2, 3$. Reliance on the computer can probably be replaced with theoretical arguments by justifying the applicability of results in [**47**, Ch. 14], [**36**, §6], [**2**, §5], and [**3**, §3] to our circumstances, but it seems less time-consuming to use the computer.

Terminology. A connected semisimple group G over a field k is *absolutely simple* if $G \neq 1$ and $G_{\overline{k}}$ has no nontrivial smooth connected proper normal subgroup. This is equivalent to irreducibility of the root system of $G_{\overline{k}}$. In the literature there is a plethora of terminology for this concept: *absolutely almost simple*, *absolutely quasi-simple*, etc. (see [**26**, §1, p. 264]).

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2. Quadratic spaces and quadratic lattices

A *quadratic space* over a ring R is a pair (M, q) consisting of a locally free R-module M of finite rank $n > 0$ equipped with an R-valued *quadratic form* on M: a map

$$q : M \to R$$

such that (i) $q(cx) = c^2 q(x)$ for all $x \in M$, $c \in R$ and (ii) the symmetric pairing

$$B_q : M \times M \to R$$

defined by $(x, y) \mapsto q(x+y) - q(x) - q(y)$ is R-bilinear. (For our purposes, the quadratic spaces of most interest will be over fields and Dedekind domains.)

For a quadratic space (M, q) over R such that M admits an R-basis $\{e_1, \ldots, e_n\}$,

$$\mathrm{disc}(q) := \det(B_q(e_i, e_j)) \in R \tag{2.1}$$

changes by $(R^\times)^2$-scaling when we change the basis. For $R = \mathbf{Z}$, this is a well-defined element of $\mathbf{Z}$ called the *discriminant* of (M, q). (For general R, the ideal $\mathrm{disc}(q)$ generates in R is independent of $\{e_i\}$ and thus globalizes to a locally principal ideal of R when M is not assumed to be free. If $R = \mathbf{Z}$ then this ideal provides less information than the discriminant in $\mathbf{Z}$.)

A *quadratic lattice* is a quadratic space (M, q) over $\mathbf{Z}$ such that $\mathrm{disc}(q) \neq 0$. For such pairs, $(M_{\mathbf{R}}, q_{\mathbf{R}})$ is a non-degenerate quadratic space over $\mathbf{R}$ and so has a *signature* (r, s) with $s = n - r$.