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GOOD GROSSHANS FILTRATION IN A FAMILY

by

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Abstract. – We reprove the main result of our joint work [**19**], with the base field replaced by a commutative Noetherian ring $\mathbf{k}$. This has repercussions for the cohomology $H^*(G, A)$ of a reductive group scheme G over $\mathbf{k}$, with coefficients in a finitely generated commutative $\mathbf{k}$-algebra A. For clarity we follow [**19**] closely.

***Résumé* (Faisceaux munis de filtration de Grosshans bonne).** – Nous généralisons le résultat principal de [**19**], en remplaçant le corps de base par un anneau commutatif noethérien $\mathbf{k}$. Ainsi on obtient de l'information sur la cohomologie $H^*(G, A)$, où G est un schéma en groupes réductif sur $\mathbf{k}$ et A est une $\mathbf{k}$-algèbre de type fini. Nous suivons les grandes lignes du texte original [**19**].

1. Introduction

Let $\mathbf{k}$ be a Noetherian ring. Consider a flat linear algebraic group scheme G defined over $\mathbf{k}$. Recall that G has the cohomological finite generation property (CFG) if the following holds: Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally by $\mathbf{k}$-algebra automorphisms. (So G acts from the right on $\mathrm{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a $\mathbf{k}$-algebra. Here, as in [**12**, I.4], we use the cohomology introduced by Hochschild, also known as 'rational cohomology'.

This note is part of the project of studying (CFG) for reductive G. More specifically, the intent of this note is to generalize the main result of [**19**] to the case where the base ring of GL_N is our Noetherian ring $\mathbf{k}$. That will allow to enlarge the scope of several results in [**20**], [**6**]. Let us give an example. Let G be a reductive group scheme over $\mathrm{Spec}(\mathbf{k})$ in the sense of SGA3. Recall this means that G is affine and smooth over $\mathrm{Spec}(\mathbf{k})$ with geometric fibers that are connected reductive. Let G act rationally by $\mathbf{k}$-algebra automorphisms on a finitely generated commutative $\mathbf{k}$-algebra A. We do

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not know (CFG) in this generality, but now we can state at least that the $H^m(G,A)$ are Noetherian modules over the ring of invariants A^G. And if $\mathbf{k}$ contains a finite ring we do indeed know that $H^*(G,A)$ is a finitely generated $\mathbf{k}$-algebra. See Section 10 for these results and related material.

To formulate the main result, let $N \geq 1$ and let G be the affine algebraic group GL_N or SL_N over $\mathbf{k}$. We use notations and terminology as in [**19**], [**6**]. Recall in particular that a G-module V module is said to have *good Grosshans filtration* if the embedding $\operatorname{gr} V \to \operatorname{hull}_\nabla(\operatorname{gr} V)$ of Grosshans is an isomorphism [**6**, Definition 27]. Such a module is G-acyclic. It does not need to be flat over $\mathbf{k}$. The module V has a good Grosshans filtration if and only if it satisfies the following cohomological criterion: $H^i(G, V \otimes_{\mathbf{k}} \nabla(\lambda))$ vanishes for all $i > 0$ and all dominant weights λ. Over fields this is the familiar criterion for having a good filtration. Indeed over a field there is no difference between 'good filtration' and 'good Grosshans filtration'. But modules with good filtration are required to be free over $\mathbf{k}$ and this is not the right requirement in our present setting. We wish to allow the filtration of V to have an associated graded that is a direct sum of modules of the form $\nabla(\lambda) \otimes_{\mathbf{k}} J(\lambda)$ with G acting trivially on $J(\lambda)$. The $J(\lambda)$ do not have to be free over $\mathbf{k}$; they even do not have to be flat over $\mathbf{k}$.

Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally by $\mathbf{k}$-algebra automorphisms. Let M be a Noetherian A-module on which G acts compatibly. This means that the structure map $A \otimes_{\mathbf{k}} M \to M$ is a G-module map. We also say that M is a (Noetherian) AG-module. (Later our convention will be that any AG-module is Noetherian.)

Our main theorem is

Theorem 1.1. – *If A has a good Grosshans filtration, then there is a finite resolution*

$$0 \to M \to N_0 \to N_1 \to \cdots \to N_d \to 0$$

where the N_i are Noetherian AG-modules with good Grosshans filtration.

Corollary 1.2. – *The $H^i(G,M)$ are Noetherian A^G-modules and they vanish for $i \gg 0$.*

Proof. – One may compute $H^*(G,M)$ with the resolution $N_0 \to \cdots N_d \to 0$. So the result follows from invariant theory [**6**, Theorem 12, Theorem 9]. □

Remark 1.3. – It is natural to ask if the same results hold for other Dynkin types. For the Corollary the answer is yes, because of Theorem 10.5 below. For Theorem 1.1 we do not know how to keep the N_i Noetherian, but otherwise it goes through by [**6**, Proposition 28] and Theorem 10.5 below.

We will actually prove a more technical version of the theorem. This is the key difference with the proof in [**19**]. Recall that the fundamental weights $\varpi_1, \ldots, \varpi_N$ are given by $\varpi_i = \sum_{j=1}^i \epsilon_j$. Let ρ be their sum and let $\mathrm{St}_r = \nabla(r\rho)$. Let U be the subgroup of unipotent upper triangular matrices.

Proposition 1.4. – *If A has a good Grosshans filtration, then*

– $H^i(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ *vanishes for* $i \gg 0$,
– $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathrm{St}_r \otimes_{\mathbf{k}} \mathrm{St}_r \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ *vanishes for* $r \gg 0$.

Define the 'Grosshans filtration dimension' of a nonzero M to be the minimum d for which $H^{d+1}(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes. As $(\mathrm{St}_r \otimes_{\mathbf{k}} \mathrm{St}_r)^G = \mathbf{k}$, we have a natural map $V \to V \otimes_{\mathbf{k}} \mathrm{St}_r \otimes_{\mathbf{k}} \mathrm{St}_r$ for any G-module V. In the theorem one may start with $N_0 := M \otimes_{\mathbf{k}} \mathrm{St}_r \otimes_{\mathbf{k}} \mathrm{St}_r$. The cokernel of $M \to N_0$ will then have a lower Grosshans filtration dimension. And Grosshans filtration dimension zero implies good Grosshans filtration [**6**, Proposition 28].

Remark 1.5. – In Proposition 1.4 it would suffice to tensor once with St_r. Our formulation is adapted to the proof of Theorem 1.1.

As in [**19**] the method of proof of Theorem 1.1 is based on the functorial resolution [**16**] of the ideal of the diagonal in $Z \times Z$ when Z is a Grassmannian of subspaces of $\mathbf{k}^N$. This is used inductively to study equivariant sheaves on a product X of such Grassmannians. That leads to a special case of the theorems, with A equal to the Cox ring of X, multigraded by the Picard group $\mathrm{Pic}(X)$, and M compatibly multigraded. Next one treats cases when on the same A the multigrading is replaced with a 'collapsed' grading with smaller value group and M is only required to be multigraded compatibly with this new grading. Here the trick is that an associated graded of M has a multigrading that is collapsed a little less. The suitably multigraded Cox rings are then used as in [**19**] to cover the general case 1.1.

Recall that Section 10 gives some consequences for earlier work.

2. Recollections and conventions

Some unexplained notations, terminology, properties, ... can be found in [**12**]. Until Section 8 the group G is either GL_N or SL_N. Some things are best told with GL_N, but the conclusion of Proposition 1.4 refers only to the SL_N-module structure.

First let $G = \mathrm{GL}_N$, with B^+ its subgroup of upper triangular matrices, B^- the opposite Borel subgroup, $T = B^+ \cap B^-$ the diagonal subgroup, $U = U^+$ the unipotent radical of B^+. The roots of U are positive, contrary to the Århus convention followed in [**6**]. The character group $X(T)$ has a basis $\epsilon_1 \dots, \epsilon_N$ with $\epsilon_i(\mathrm{diag}(t_1, \dots, t_N)) = t_i$. An element $\lambda = \sum_i \lambda_i \epsilon_i$ of $X(T)$ is often denoted $(\lambda_1, \dots, \lambda_N)$. It is called a polynomial weight if the λ_i are nonnegative. It is called a dominant weight if $\lambda_1 \geq \cdots \geq \lambda_N$. It is called anti-dominant if $\lambda_1 \leq \cdots \leq \lambda_N$. The fundamental weights $\varpi_1, \dots, \varpi_N$ are given by $\varpi_i = \sum_{j=1}^{i} \epsilon_j$. If $\lambda \in X(T)$ is dominant, then $\mathrm{ind}_{B^-}^G(\lambda)$ is the *dual Weyl module* or *costandard module* $\nabla_G(\lambda)$, or simply $\nabla(\lambda)$, with highest weight λ. The *Grosshans height* of λ is $\mathrm{ht}(\lambda) = \sum_i (N - 2i + 1)\lambda_i$. It extends to a homomorphism $\mathrm{ht} : X(T) \otimes \mathbb{Q} \to \mathbb{Q}$. The determinant representation has weight ϖ_N and one has $\mathrm{ht}(\varpi_N) = 0$. Each positive root β has $\mathrm{ht}(\beta) > 0$. If λ is a dominant polynomial weight, then $\nabla_G(\lambda)$ is called a *Schur module*. If α is a partition with at most N parts then we may view it as a dominant polynomial weight and the *Schur functor* S^α maps $\nabla_G(\varpi_1)$

to $\nabla_G(\alpha)$. (This is the convention followed in [**16**]. In [**1**] the same Schur functor is labeled with the conjugate partition $\tilde{\alpha}$.) The formula $\nabla(\lambda) = \mathrm{ind}_{B^-}^G(\lambda)$ just means that $\nabla(\lambda)$ is obtained from the Borel-Weil construction: $\nabla(\lambda)$ equals $H^0(G/B^-, \mathscr{L}_\lambda)$ for a certain line bundle $\mathscr{L}_\lambda$ on the flag variety G/B^-.

Now consider the case $G = \mathrm{SL}_N$. There are similar conventions for SL_N-modules. For instance, the costandard modules for SL_N are the restrictions of those for GL_N. The Grosshans height on $X(T)$ induces one on $X(T \cap \mathrm{SL}_N) \otimes \mathbb{Q}$. The *multicone* $\mathbf{k}[\mathrm{SL}_N/U]$ consists of the f in the coordinate ring $\mathbf{k}[\mathrm{SL}_N]$ that satisfy $f(xu) = f(x)$ for $u \in U$. As an SL_N-module it is the direct sum of all costandard modules. It is also a finitely generated algebra [**15**], [**8**], [**6**, Lemma 23]. Note that $\mathbf{k}[\mathrm{SL}_N/U]$ is SL_N-equivariantly isomorphic to $\mathbf{k}[\mathrm{SL}_N/U^-]$, so that here it does not matter whether one follows the Århus convention or not.

Definition 2.1. – A *good filtration* of a G-module V is a filtration $0 = V_{\leq -1} \subseteq V_{\leq 0} \subseteq V_{\leq 1} \subseteq \cdots$ by G-submodules $V_{\leq i}$ with $V = \bigcup_i V_{\leq i}$, so that its associated graded $\mathrm{gr}\, V$ is a direct sum of costandard modules.

A Schur filtration of a polynomial GL_N-module V is a filtration $0 = V_{\leq -1} \subseteq V_{\leq 0} \subseteq V_{\leq 1} \subseteq \cdots$ by GL_N-submodules with $V = \bigcup_i V_{\leq i}$, so that its associated graded $\mathrm{gr}\, V$ is a direct sum of Schur modules. The *Grosshans filtration* of V is the filtration with $V_{\leq i}$ the largest G-submodule of V whose weights λ all satisfy $\mathrm{ht}(\lambda) \leq i$. Good filtrations and Grosshans filtrations for SL_N-modules are defined similarly. The literature contains more restrictive definitions of good/Schur filtrations. Ours are the right ones when dealing with representations that need not be finitely generated over $\mathbf{k}$.

Let M be a G-module provided with the Grosshans filtration. Recall from [**6**] that M has *good Grosshans filtration* if the embedding of $\mathrm{gr}\, M$ into $\mathrm{hull}_\nabla(\mathrm{gr}\, M) = \mathrm{ind}_{B^-}^G M^U$ is an isomorphism. Then $\mathrm{gr}\, M$ is a direct sum of modules of the form $\nabla(\lambda) \otimes_{\mathbf{k}} J(\lambda)$ with G acting trivially on $J(\lambda)$. The $J(\lambda)$ need not be flat. If they are all free then we are back at the case of a good filtration.

A G-module M has good Grosshans filtration if and only if $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes [**6**, Proposition 28]. And $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes if and only if $H^1(\mathrm{SL}_N, M \otimes_{\mathbf{k}} V)$ vanishes for every module V with good filtration. A module with good filtration has good Grosshans filtration and is flat as a $\mathbf{k}$-module. The tensor product of two modules with good filtration has good filtration [**12**, Lemma B.9, II Proposition 4.21]. The tensor product of a module with good filtration and one with good Grosshans filtration thus has good Grosshans filtration. If M is a G-module, then $M \otimes \mathbf{k}[G]$ has a good Grosshans filtration by [**12**, I Lemma 4.7a]. This may be used in dimension shift arguments. If $H^i(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$ vanishes, then so does $H^{i+1}(\mathrm{SL}_N, M \otimes_{\mathbf{k}} \mathbf{k}[\mathrm{SL}_N/U])$. This follows from [**6**, Proposition 28] and dimension shift. The following Lemma is also proved by dimension shift.

Lemma 2.2. – *If M has finite Grosshans filtration dimension d and V has good filtration, then $M \otimes V$ has finite Grosshans filtration dimension $\leq d$.*