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ADJOINT QUOTIENTS OF REDUCTIVE GROUPS

by

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Abstract. – Let G be a reductive group over a commutative ring k. In this article, we prove that the adjoint quotient $\mathrm{G} /\!/ \mathrm{G}$ is stable under base change. Moreover, if G has a maximal torus T, then the adjoint quotient of the torus T by its Weyl group will be isomorphic to $\mathrm{G} /\!/ \mathrm{G}$. Then we focus on the semisimple simply connected group G of the constant type. In this case, $\mathrm{G} /\!/ \mathrm{G}$ is isomorphic to the Weil restriction $\prod_{\mathrm{D}/\mathrm{spec}\, k} \mathbb{A}^1_{\mathrm{D}}$, where D is the Dynkin scheme of G. Then we prove that for such G, the Steinberg's cross-section can be defined over k if G is quasi-split and without A_{2m}-type components.

Résumé **(Quotients adjoints de groupes réductifs).** – Soit k un anneau commutatif et G un groupe réductif sur k. Dans cet article, on va definir le quotient adjoint $\mathrm{G} /\!/ \mathrm{G}$ de G sur k et démontrer que la construction est stable par changement de base. En plus, si G possède un tore maximal T, le quotient adjoint de T par son groupe de Weyl est isomorphe à $\mathrm{G} /\!/ \mathrm{G}$. Dans la derniere section, on se concentre sur le cas G semi-simple simplement connexe de type constant. Dans ce cas, $\mathrm{G} /\!/ \mathrm{G}$ est isomorphe à la restriction de Weil $\prod_{\mathrm{D}/\mathrm{spec}\, k} \mathbb{A}^1_{\mathrm{D}}$, où D est le schéma de Dynkin. Si G est de plus quasi-déployable et sans composantes de type A_{2m}, on peut construire la cross-section de Steinberg sur k.

1. Introduction

Let k be a commutative ring and G be a reductive group over k. In this article, we want to discuss the adjoint quotient of G which is denoted by $\mathrm{G} /\!/ \mathrm{G}$.

Roughly speaking, the adjoint quotient of G is determined by those regular functions of G which are constants on the conjugacy classes of G. Suppose that G contains a maximal torus T and let W be the corresponding Weyl group. Then the G-conjugation action on the regular functions of G induces a W-conjugation action on the

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regular functions of T. Let T // W be the adjoint quotient of T by W. The natural restriction on regular functions induces a natural morphism

$$\iota : \mathrm{T} /\!/ \mathrm{W} \to \mathrm{G} /\!/ \mathrm{G}.$$

When k is an algebraically closed field, $i^\sharp$ is an isomorphism. One can find the classical treatment about adjoint quotients over algebraically closed fields in Steinberg's paper [**12**] §6, or in Humphreys's book [**7**] Chap. 3.

In this article, we will show that the same result holds for any commutative ring k. Namely, we will prove the following theorem:

Theorem. – *Let k be a commutative ring and* G *be a reductive group defined over k. Suppose that* G *contains a maximal k-torus* T. *Let* W *be the corresponding Weyl group of* T. *Then* $\mathrm{T} /\!/ \mathrm{W} \xrightarrow{\sim} \mathrm{G} /\!/ \mathrm{G}$.

The strategy we take here is actually the same one used for k an algebraically closed field. In § 3, we will prove the Theorem over $\mathbb{Z}$ and generalize the result to arbitrary commutative rings in § 4.

In § 5, we focus on the adjoint quotient of the semisimple simply connected group scheme of constant type. For such group G, G // G is isomorphic to the Weil restriction $\prod_{\mathrm{D}/\mathrm{Spec}k} \mathbb{A}^1_{\mathrm{D}}$, where D is the Dynkin scheme of G. Moreover, we prove that the Steinberg's cross-section ([**12**], Thm. 1.4) can be defined over arbitrary commutative ring k if G is quasi-split and without A_{2m}-type components. At the end of the article, we prove that for a semisimple simply connected group scheme G of constant type, it always contains a semisimple regular element over a semi-local ring k, and the centralizer of a semisimple regular element is a maximal torus of G.

2. Notations and Definitions

Let k be a commutative ring. For an affine k-scheme X, we let $k[\mathrm{X}]$ be the ring of regular functions of X. For a k-scheme X, a k-algebra A, we let X_A be the fiber product $\mathrm{X} \times_k \mathrm{Spec}\mathrm{A}$. Let $\mathbb{G}_\mathrm{m}$ (resp. $\mathbb{G}_\mathrm{a}$) be the multiplicative (resp. additive) group defined over $\mathbb{Z}$.

For a k-module V, we regard it as a functor by defining

$$\mathrm{V}(\mathrm{A}) = \mathrm{V} \otimes_k \mathrm{A}, \text{ for all } k\text{-algebras A}.$$

In order to define the adjoint quotient of G over an arbitrary commutative ring k, we first define a G-conjugation action on the k-module $\mathrm{V} := k[\mathrm{G}]$. Let A be a k-algebra and $g \in \mathrm{G}(\mathrm{A})$, $f \in \mathrm{V}(\mathrm{A})$, we define

$$(g.f)(x) = f(g^{-1}xg), \text{ for all A-algebras A}', \text{ and for all } x \in \mathrm{G}(\mathrm{A}').$$

Let $c : \mathrm{V} \to k[\mathrm{G}] \otimes \mathrm{V}$ be the comodule map corresponding to the conjugation action defined above. We define

$$k[\mathrm{G}]^{\mathrm{G}} := \{f \in \mathrm{V}(k) |\ \sigma.f_{\mathrm{A}} = f_{\mathrm{A}},\ \forall\ \sigma \in \mathrm{G}(\mathrm{A}),\ \forall\ k-\mathit{algebras}\ \mathrm{A}\},$$

Let $\mathrm{G} \mathbin{/\!/} \mathrm{G} = \mathrm{Spec}(k[\mathrm{G}]^{\mathrm{G}})$ be the *adjoint quotient* of G. Suppose that G contains a maximal torus T and let W be the corresponding Weyl group. Then the G-conjugation action on $k[\mathrm{G}]$ induces a W-conjugation action on $k[\mathrm{T}]$ and let $\mathrm{T} \mathbin{/\!/} \mathrm{W} = \mathrm{Spec}(k[\mathrm{T}]^{\mathrm{W}})$. The natural restriction from $k[\mathrm{G}]$ to $k[\mathrm{T}]$ induces a natural homomorphism

$$i : k[\mathrm{G}]^{\mathrm{G}} \to k[\mathrm{T}]^{\mathrm{W}}.$$

When k is an algebraically closed field, i is an isomorphism. Namely,

Theorem 2.1. – *Let k be an algebraically closed field. Let* G *be a semisimple k-group and* T *be a maximal torus of* G. *Let* W *be the Weyl group with respect to* T. *Then the restriction map $i : k[\mathrm{G}]^{\mathrm{G}} \to k[\mathrm{T}]^{\mathrm{W}}$ is an isomorphism. Furthermore, if* G *is semisimple simply connected, then $k[\mathrm{G}]^{\mathrm{G}}$ is freely generated as a commutative k-algebra by the characters of the irreducible representations with respect to the fundamental highest weights.*

Proof. – The injectivity relies on the fact that the semisimple regular elements in G form an open dense subset.

The idea to prove the surjectivity is to find a set of representations $\rho : \mathrm{G} \to \mathrm{GL}_{n,k}$ such that the corresponding set of characters restricted to T generates $k[\mathrm{T}]^{\mathrm{W}}$. For more details, one can refer to [**7**] 3.2, [**12**] §6 and [**8**] Part II, 2.6. □

In the following section, we want to repeat some arguments in the standard proof to show that those techniques fit quite well for reductive groups. Moreover, we will generalize these arguments from fields to $\mathbb{Z}$ when G is a split reductive group scheme over $\mathbb{Z}$, and T is a maximal $\mathbb{Z}$-torus of G.

3. The adjoint quotient over $\mathbb{Z}$

In this section, we will show that a result similar to Theorem 2.1 also holds over $\mathbb{Z}$. Namely,

Theorem 3.1. – *Let* G *be a split reductive $\mathbb{Z}$-group and* T *be a maximal $\mathbb{Z}$-torus of* G. *Let* W *be the Weyl group with respect to* T. *Then the restriction map $i : \mathbb{Z}[\mathrm{G}]^{\mathrm{G}} \to \mathbb{Z}[\mathrm{T}]^{\mathrm{W}}$ is an isomorphism.*

As the first step, we want to generalize the techniques used to prove Theorem 2.1.

3.1. The W-conjugation action on tori. – Let k be a commutative ring and T be a split torus over k. Let M be the character group of T which can be regarded as an additive group, and $\mathrm{M}^\vee$ be the dual of M considered as $\mathbb{Z}$-module. Let $\mathscr{R} = (\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee)$ be a reduced root datum with respect to T and W be the corresponding Weyl group. Let Π be a system of simple roots of R. Let M^+ be the set of characters λ which satisfy $(\alpha^\vee, \lambda) \geq 0$ for all $\alpha^\vee \in \Pi^\vee$. A character λ is called *dominant* if $\lambda \in \mathrm{M}^+$.

Here we want to look at $k[\mathrm{T}]^{\mathrm{W}}$ more closely. Since $k[\mathrm{T}] = k[\mathrm{M}]$ and W permutes M, we observe that $k[\mathrm{T}]$ is a W-permutation module under the conjugation action. Let e^λ be the element in $k[\mathrm{T}]$ corresponding to λ in M. Then $k[\mathrm{T}]^{\mathrm{W}}$ is generated by elements of the form $\mathrm{Sym}(\mathrm{e}^\lambda) := \sum_{w \in \mathrm{W}/\mathrm{W}_\lambda} \mathrm{e}^{w(\lambda)}$, where $\lambda \in \mathrm{M}$ and W_λ is the stabilizer of λ. Since for each $\lambda \in \mathrm{M}$ we can find a $w \in \mathrm{W}$ such that $w\lambda$ is in M^+, $k[\mathrm{T}]^{\mathrm{W}}$ is a free k-module generated by the set $\{\mathrm{Sym}(\mathrm{e}^\lambda) | \ \lambda \in \mathrm{M}^+\}$ (ref. [**3**], Chap. VI, §3, Lemma 3), which in turn means that $k[\mathrm{T}]^{\mathrm{W}}$ is determined by the W-action on M and therefore is stable under arbitrary base change. We rephrase this fact as a lemma:

Lemma 3.2. – *Let k,* T *and* W *be defined as above. Then the ring $k[\mathrm{T}]^{\mathrm{W}}$ is a free k-module generated by the set $\{\mathrm{Sym}(\mathrm{e}^\lambda) \mid \lambda \in \mathrm{M}^+\}$ and hence is determined by the* W*-action on* M. *In particular, we have $k[\mathrm{T}]^{\mathrm{W}} \otimes k' = k'[\mathrm{T}]^{\mathrm{W}}$, for any k-algebra k'.*

However, besides the basis $\{\mathrm{Sym}(\mathrm{e}^\lambda) | \ \lambda \in \mathrm{M}^+\}$, we sometimes need to choose an alternative basis to simplify our proof. The next two lemmas are useful for this purpose:

Lemma 3.3. – *Let I be an ordered set satisfying the following condition:*

(Min) *Each nonempty subset of* I *contains a minimal element.*

Let A *be a commutative ring, E be an* A*-module, and $\{e_i\}_{i \in I}$ be a basis of E. Let $\{x_i\}_{i \in I}$ be a family of elements such that $x_i = e_i + \sum_{j<i} a_{i,j}\ e_j$, where $a_{i,j} \in \mathrm{A}$ and only finitely many $a_{i,j}$ are nonzero. Then also $\{x_i\}_{i \in \mathrm{I}}$ is a basis of E.*

Proof. – [**3**], Chap. VI, §3, Lemma 4. □

Let us define a partial order on M with respect to Π by $\lambda \geq 0$ if $\lambda = \sum_{s \in \Pi} a_s s$ where a_s's are nonnegative integers.

Lemma 3.4. – *Given an element $\lambda \in \mathrm{M}^+$, the set $\mathrm{I}(\lambda) := \{\mu \in \mathrm{M}^+ \mid \mu \leq \lambda\}$ is finite.*

Proof. – If the root datum $\mathscr{R}$ is semisimple, then one can find a proof of the above lemma in [**3**], Chap. VI, §3, Prop. 3. For the reduced root datum $\mathscr{R}$, let $\mathrm{corad}(\mathscr{R})$ be the coradical of $\mathscr{R}$, and $\mathrm{ss}(\mathscr{R})$ be the semisimple part of $\mathscr{R}$ ([**5**], Exp. XXI, 6.3.2, 6.5.4, and 6.5.5). The partial order on $\mathscr{R}$ induces a partial order on $\mathrm{ss}(\mathscr{R})$, and we define a partial order on $\mathrm{corad}(\mathscr{R})$ as $x \leq y$ iff $x = y$. Let p be the canonical isogeny $p : \mathrm{corad}(\mathscr{R}) \times \mathrm{ss}(\mathscr{R}) \to \mathscr{R}$, and d be the degree of the isogeny. If $\mu \leq \lambda$ in $\mathscr{R}$, then $d\mu \leq d\lambda$ in $\mathrm{corad}(\mathscr{R}) \times \mathrm{ss}(\mathscr{R})$. Hence we reduce the case to the semisimple case, and the lemma follows. □