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Simon Pepin Lehalleur

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SUBGROUPS OF MAXIMAL RANK OF REDUCTIVE GROUPS

by

Simon Pepin Lehalleur

Abstract. – In the paper **[2]**, Borel and De Siebenthal study the structure of subgroups of maximal rank of compact Lie groups. In this note, we show how the methods of **[6]** allow an extension of their results to reductive group schemes over general bases. We discuss in particular the exceptional subgroups that occur in characteristics 2 and 3.

***Résumé* (Sous-groupes de groupes réductifs de rang maximal).** – Dans l'article **[2]**, Borel et De Siebenthal étudient la structure des sous-groupes de rang maximal des groupes de Lie compacts. Dans cette note, nous montrons comment les méthodes de **[6]** permettent d'étendre leurs résultats aux schémas en groupes réductifs sur une base génerale. Nous discutons en particulier les sous-groupes exceptionnels qui apparaissent en charactéristiques 2 et 3.

All the references of the form Exp. are to **[6]** (in the new edition).

Introduction

In *Les sous-groupes fermés des groupes de Lie clos* **[2]**, Borel and De Siebenthal prove the following

***Theorem 0.1* ([2], Théorème 5).** – *Let G be a compact Lie group, H a closed connected subgroup of G.*
Assume that H has maximal rank*, i.e., that it contains a maximal (compact) torus of G. Then*

$$H = \mathrm{Centr}_G(\mathrm{Centr}(H))^\circ.$$

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In particular, H is determined by its center. The discreteness of groups of automorphisms of tori implies that one can replace the centralizer by the normalizer in this equality.

Let S be a base scheme, and G a reductive S-group scheme. We make the following definition (which is the special case of Exp. XXII, Définition 5.2.1 with G reductive):

Definition 0.2. – A sub-S-group scheme H of G is a subgroup of maximal rank if:

(i) H is smooth of finite presentation with connected fibers.

(ii) For all $\bar{s} \to S$ geometric point of S, $H_{\bar{s}}$ contains a maximal torus of $G_{\bar{s}}$.

From Exp. XII Théorème 1.7 b) and the fact that the reductive rank of H is locally constant (because it is equal to the one of G) we know that H admits maximal tori locally for the étale topology, and such maximal tori are also maximal tori of G. Hence G and H share a maximal torus locally for the étale topology.

Note that we did not suppose H closed. This is the case when H is reductive (as any reductive subgroup of a separated group scheme of finite presentation is closed, see [**4**], Theorem 5.3.5). We will give a proof that H is closed in general, see the end of Section 3.

Examples of subgroups of maximal rank in reductive S-groups include maximal tori, Borel subgroups, more general parabolic subgroups and their Levi subgroups. The more interesting cases of the Borel-De Siebenthal theorem are the semi-simple subgroups of maximal rank. In classical groups, those are well known: a typical example in type D_n is, for (V, q) a quadratic space of even dimension $2n$ and W a non-degenerate subspace of even dimension $2k$, the closed immersion $\mathbb{SO}(W) \times \mathbb{SO}(W^{\perp}) \hookrightarrow \mathbb{SO}(V)$ (Notice that $\operatorname{rank}(\mathbb{SO}_n) = \lfloor \frac{n}{2} \rfloor$ for all $n \geq 2$, so that the rank of $\mathbb{SO}(W) \times \mathbb{SO}(W^{bot})$ is equal to the rank of $\mathbb{SO}(V)$ if and only if W is odd dimensional). So the main interest of the theorem lies in exceptional groups.

One can derive from Theorem 0.1 and consideration on Cartan involutions in reductive algebraic groups over $\mathbb{C}$ the following

Proposition 0.3. – *Let G be an reductive algebraic group over an algebraically closed field of characteristic zero. Let H be a reductive subgroup of G of maximal rank. Then:*

$$H = \mathrm{Centr}_G(\mathrm{Centr}(H))^{\circ}.$$

We do not detail the argument since we will prove a generalization of this result later.

Let us call H a (reductive) subsystem subgroup if it satisfies the conclusion of the previous proposition. Our primary goal is to characterize those subgroups. It turns out there are reductive subgroups of maximal rank which are not subsystem subgroups and which provide counterexamples to the direct generalization of the previous proposition: more precisely, there are exotic counterexamples over fields of characteristic 2 and 3 (and more generally bases where 2 or 3 are locally 0) for each non simply-laced type. See 2.2 and 3.1 for a precise statement.

Let us now describe the contents of this article.

In Section 1 and 2, we study the Lie algebras of subgroups of maximal rank and relate them to certain subsets of root systems.

In Section 3, we prove (following closely a method of Demazure) that in the split case (with both G and H split), the subsystem subgroups (resp. the exotic subgroups) do exist in the Chevalley groups over $\mathrm{Spec}(\mathbb{Z})$ (resp. $\mathrm{Spec}(\mathbb{F}_2)$ and $\mathrm{Spec}(\mathbb{F}_3)$).

In Section 4, we explain how to describe subgroups of maximal rank in general reductive group schemes in terms of those in semi-simple simply connected group schemes with absolutely simple fibers.

In Section 5, we explain how certain exotic subgroups can be related to the subsystem case by very special isogenies in characteristics 2 and 3.

In Section 6, we extend the Borel-De Siebenthal algorithm to subsystem subgroups in arbitrary charateristics. The original paper **[2]** uses Theorem 0.1 to deduce a classification algorithm for the subgroups of maximal rank. Indeed, the theorem implies that the *maximal* (closed connected) subgroups of maximal rank are centralizers of elements of finite order. Conjugacy classes of these were classified by Cartan leading to a type by type description in terms of extended Dynkin diagrams. The general subgroup of maximal rank is then obtained recursively. To extend this to arbitrary characteristics, the key idea, due to Serre **[14]**, is that the correct analogue of an element of finite order is a closed immersion of μ_n into G. With this notion, many standard results on centralizers of semi-simple elements extend without restrictions on the characteristics.

The appendix consists of a list of the irreducible extended Dynkin diagrams with some additional data relevant to the results in Section 6.

We do not consider here the applications of the Borel-De Siebenthal theorem. Let us just give references to a sample of these for the curious reader. A reformulation of the theorem in terms of reductive Lie groups allows one to classify the so-called *equirank* semi-simple Lie groups (those which share a maximal compact torus with their maximal compact subgroups), which play a fondamental role in harmonic analysis on symmetric spaces and in Harish Chandra's approach to representation theory: see **[13]**, **[15]**. The subgroups of maximal rank (including the exotic ones) play a small role in the classification of all maximal subgroups of positive dimension of simple groups over algebraically closed fields in arbitrary characteristic, which was recently completed by Liebeck and Seitz: see **[12]** and the references there. Finally, the subsystem subgroups have been used to study the Galois cohomology of exceptional groups: see **[7]**, **[8]**.

Part of what we present could be done for subgroups of maximal rank in S-groups of type (RR), as in Exp. XXII. We stick to the reductive case for simplicity.

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1. Lie algebras of subgroups of maximal rank

Let H be a subgroup of maximal rank of a reductive S-group G. Let us assume that there exists a split maximal torus $T \subset H \subset G$, that G is split over S, and let us choose a splitting $G = (G, T, M, R)/S$ (cf. Exp. XXII, Définition 1.13). This applies étale-locally on S, or after base change to a geometric point. Then $\mathfrak{h} = \mathrm{Lie}(H)$ is a T-invariant (hence locally a direct factor) sub-bundle of $\mathfrak{g} = \mathrm{Lie}(G)$, containing $\mathfrak{g}^0 = \mathrm{Lie}(T)$, and thus takes the form

$$\mathfrak{h} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in R'} \mathfrak{g}^\alpha \tag{1}$$

where R' is a Zariski locally constant subset of R. When H is itself reductive, it is clear from this equality that Zariski locally on S, (H, T, M, R') is a splitting of H. This certainly implies that R' is symmetric, i.e., $R' = -R'$. In Exp. XXII, Proposition 5.10.1, it is proved conversely that if R' is symmetric, H is reductive.

It can happen that H is not determined by its Lie algebra. For example, all maximal tori in SL_2 over a field of characteristic 2 have the same Lie algebra. We will see however in the proof of the main theorem in Section 3 that in most cases (in particular if 2 is everywhere non zero on S or if G is adjoint) one can reconstruct H from its Lie algebra.

Assume for the rest of this Section that H is reductive. By Exp. XXII, Corollaire 4.17, $\mathrm{Centr}(H)$ is representable by a closed subgroup of H of multiplicative type (it is also the reductive center of H, i.e., the "intersection of its maximal tori"). By Exp. XI, Corollaire 5.3, the centralizer of the subgroup of multiplicative type $\mathrm{Centr}(H)$ in G is representable by a closed smooth sub-S-group $\mathrm{Centr}_G(\mathrm{Centr}(H))$ of G. Finally, by Exp. VI_B, Corollaire 4.4, the collection of connected components of the fibers of this smooth subgroup is represented by a smooth sub-S-group with connected fibers $\mathrm{Centr}_G(\mathrm{Centr}(H))^\circ$. Hence the right-hand side of the Borel-De Siebenthal theorem is well defined in our context. It certainly contains H (using the connexity of H), and the question is whether there is equality.

Proposition 1.1. – *Let G, H be as above. The two following conditions are equivalent:*

1. $H = \mathrm{Centr}_G(\mathrm{Centr}(H))^\circ$.
2. *For all $\bar{s} \to S$ geometric point and for a maximal torus $T \subset H_{\bar{s}}$, write the decomposition (1). Then $R' = \mathbb{Z}R' \cap R \subset M$.*

Proof. – We have to test whether the map $i : H \subset \mathrm{Centr}_G(\mathrm{Centr}(H))^\circ$ between smooth S-groups is an isomorphism. By the fibral criterion of isomorphism ([**9**] IV_4, 17.9.5 or [**4**] Lemma B.3.1) and fppf descent, this can be tested on geometric fibers of S. We can thus assume that there is a maximal split torus T inside H and fix splittings $G = (G, T, M, R)$ and $H = (H, T, M, R')$ as above. Since we are over a field, i is automatically closed, and we are reduced to check whether it is an open immersion, or equivalently (since i is a monomorphism) that i is étale. By homogeneity, it is enough to check this at the neutral element, i.e., to check that $\mathrm{Lie}(i)$ is an isomorphism.