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GROUP SCHEMES OUT OF BIRATIONAL GROUP LAWS, NÉRON MODELS

by

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Abstract. – In this note, we present the theorem of extension of birational group laws in both settings of classical varieties (Weil) and schemes (Artin). We improve slightly the original proof and result with a more direct construction of the group extension, a discussion of its separation properties, and the systematic use of algebraic spaces. We also explain the important application to the construction of Néron models of abelian varieties. This note grew out of lectures given by Ariane Mézard and the second author at the Summer School "Schémas en groupes" held in the CIRM (Luminy) from 29 August to 9 September, 2011.

Résumé **(Schémas de groupes obtenus à partir de lois de groupe birationnelles, modèles de Néron)**
Dans cette note, nous présentons le théorème d'extension d'une loi de groupe birationnelle en un groupe algébrique, dans le cadre des variétés algébriques classiques (Weil) et des schémas (Artin). Nous améliorons légèrement le résultat original et sa preuve en donnant une construction plus directe du groupe, en apportant des compléments sur ses propriétés de séparation, et en utilisant systématiquement les espaces algébriques. Nous expliquons aussi l'application importante à la construction des modèles de Néron des variétés abéliennes. Cette note est issue des cours donnés par Ariane Mézard et le second auteur à l'École d'été « Schémas en groupes » qui s'est tenue au CIRM (Luminy) du 29 août au 9 septembre 2011.

1. Introduction

This paper is devoted to an exposition of the generalization to group schemes of Weil's theorem in [**19**] on the construction of a group from a birational group law, as can be found in Artin's Exposé XVIII in SGA3 [**2**]. In addition, we show how this theorem is used by Néron in order to produce canonical smooth models (the famous *Néron models*) of abelian varieties.

The content of Weil's theorem is to extend a given "birational group law" on a scheme X to an actual multiplication on a group scheme G birational to X. The

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original motivation of Weil was the algebraic construction of the Jacobian varieties of curves [**18**]. This construction was extended by Rosenlicht to generalized Jacobians [**15**]. Weil's ideas were later used by Demazure in his thesis in order to show the existence of split reductive groups over the ring $\mathbf{Z}$ of integers [**6**] and then by Néron in order to study minimal models of abelian varieties [**10**]. To our knowledge, these are the three main applications of the extension theorem.

The construction of split reductive groups by Demazure uses a version of Weil's theorem written by Artin, valid for flat (maybe non-smooth) finitely presented group schemes. There, the set-theoretic arguments of Weil are replaced by sheaf-theoretic arguments. The main point then is to show that Weil's procedure gives a sheaf which is representable; since this sheaf is defined as a quotient by an fppf equivalence relation, the natural sense in which it is representable is as an algebraic space (i.e., a quotient of an étale equivalence relation of schemes, see 3.17). However, at the time when Artin figured out his adaptation of Weil's result, he had not yet discovered algebraic spaces. Thus he had to resort at times to ad hoc statements; for example, his main statement (Theorem 3.7 of [**2**]) is a bit unsatisfying. Nowadays it is more natural to use the language of algebraic spaces, and this is what we shall do. As an aside, it is clear that one may as well start from a birational group law on an *algebraic space*, but we do not develop this idea.

Another feature of Artin's proof is that he constructs G let us say "in the void", and that needs a lot of verifications that moreover are not so structured. We give a more structured proof of Theorem 3.7 of [**2**]. The idea is to construct the group space G as a subfunctor of the S-functor in groups $\mathscr{R}$ that sends T to the group of T-birational maps from X_T to itself, as in Section 5.2 of Bosch-Lütkebohmert-Raynaud [**4**]. We push the construction of [**4**] a bit further: we show that $\mathscr{R}$ is a sheaf and we define G to be the subsheaf of groups generated by the image of X under a morphism that sends a in $X(T)$ to the rational left-translation by a on X_T.

One technical detail is that whereas Artin requires X to be of finite presentation, we allow it to be only locally so (that is, maybe not quasi-compact and quasi-separated). This turns out to need no modification of our proofs, and may be interesting for instance for the treatment of Néron models of semi-abelian varieties, since these fail to be of finite type.

A significant difference between [**2**] and Section 5.2 of [**4**] is that [**4**] treats descent only in Chapter 6, after the construction of groups from birational ones. So, Chapter 5 of [**4**] is more geometric and less sheaf-theoretic than [**2**]. It is a good thing to compare the two accounts. Here are some considerations.

1. In [**2**], S is arbitrary, and X/S is faithfully flat and of finite presentation, with separated fibres without embedded components. The conclusion is that G/S is an algebraic space.
2. In Theorem 5.1/5 of [**4**], the scheme S is the spectrum of a field or of a discrete valuation ring, and X/S is separated, smooth and quasi-compact, and surjective.

3. In Theorem 6.1/1 of [**4**], S is arbitrary, X/S is smooth, separated, quasi-compact. The conclusion is that G/S is a scheme. For the proof of this theorem, whose main ideas come from Raynaud [**12**], Theorem 3.7 of [**2**] is admitted, although it is also said that if S is normal, then it can be obtained as in Chapter 5 of [**4**].
4. In [**2**] the birational group law is "strict". Proposition 5.2/2 of [**4**] and [**19**] reduce, under certain conditions, the case of a birational group law to a strict one.

Let us now briefly describe what we say on the application to Néron models. While Néron's original paper was written in the old language of Weil's Foundations and quite hard to read, the book [**4**] is a modern treatment that provides all details and more on this topic. It is however quite demanding for someone who wishes to have a quick overview of the construction. In this text, we tried to show to the reader that it is in fact quite simple to see not only the skeleton but also almost all the flesh of the complete construction. Thus we bring out the main ideas of Néron to produce a model of the abelian variety one started with, endowed with a strict birational group law. Then Weil's extension theorem finishes the job. The few things that we do not prove are:

1. the decreasing of Néron's measure for the defect of smoothness under blow-up of suitable singular strata (Lemma 5.5),
2. the theorem of Weil on the extension of morphisms from smooth schemes to smooth separated group schemes (proof of Proposition 6.4).

In both cases, using these results as black boxes does not interrupt the main line of the proof, and moreover there was nothing we could add to the proofs of these facts in [**4**].

The exposition of Weil's theorem occupies Sections 2 and 3 of the paper, while the application to Néron models occupies Sections 4 to 6.

2. A case treated by André Weil

Let k be an algebraically closed field. An *algebraic variety* over k will mean a k-scheme that is locally of finite type, separated, and reduced. For such an X, we denote $X(k)$ by X itself, that is, we forget about the non-closed points. A subvariety of X is said to be *dense* if it is topologically dense.

Let, in this paragraph, G be an algebraic variety over k with an algebraic group structure. Then the graph of the multiplication map from $G\times G$ to G is a closed subvariety Γ of $G\times G\times G$; it is the set of (a,b,c) in $G\times G\times G$ such that $c=ab$. For every i and j in $\{1,2,3\}$ with $i<j$ the projection $\mathrm{pr}_{i,j}\colon\Gamma\to G\times G$ is an isomorphism, hence Γ is the graph of a morphism $f_{i,j}:=\mathrm{pr}_k\circ\mathrm{pr}_{i,j}^{-1}$, where $\{i,j,k\}=\{1,2,3\}$, from $G\times G$ to G. We have $f_{1,2}(a,b)=ab$, $f_{1,3}(a,c)=a^{-1}c$ and $f_{2,3}(b,c)=cb^{-1}$. For X a dense open subvariety of G and W a dense open subvariety of Γ contained in $X\times X\times X$, the pair (X,W) is a strict birational group law as in the following

definition. Theorem 2.11 shows in fact that each strict birational group law is in fact obtained in this way.

Definition 2.1. – Let X be an algebraic variety over k, not empty. A *strict birational group law on* X is a subvariety (locally closed, by definition) W of $X \times X \times X$, that satisfies the following conditions.

1. For every i and j in $\{1, 2, 3\}$ with $i < j$ the projection $\mathrm{pr}_{i,j} \colon W \to X \times X$ is an open immersion whose image, denoted $U_{i,j}$, is dense in $X \times X$. For each such (i, j), we let $f_{i,j} \colon U_{i,j} \to X$ be the morphism such that W is its graph. For every such (i, j) and for every $x = (x_1, x_2, x_3)$ in X^3 the condition $x \in W$ is equivalent to: $(x_i, x_j) \in U_{i,j}$ and $x_k = f_{i,j}(x_i, x_j)$, with $\{i, j, k\} = \{1, 2, 3\}$. We denote the morphism $f_{1,2} \colon U_{1,2} \to X$ by $(a, b) \mapsto ab$. Hence, for (a, b, c) in X^3 we have $(a, b, c) \in W$ if and only if $(a, b) \in U_{1,2}$ and $c = ab$.
2. For every a in X, and for every i and j in $\{1, 2, 3\}$ with $i < j$ the inverse images of $U_{i,j}$ under the morphisms (a, id_X) and $(\mathrm{id}_X, a) \colon X \to X \times X$ are dense in X (in other words, $U_{i,j} \cap (\{a\} \times X)$ is dense in $\{a\} \times X$ and $U_{i,j} \cap (X \times \{a\})$ is dense in $X \times \{a\}$).
3. For all $(a, b, c) \in X^3$ such that (a, b), (b, c), (ab, c) and (a, bc) are in $U_{1,2}$, we have $a(bc) = (ab)c$.

From now on, X is an algebraic variety over k with a strict rational group law W. The idea in what follows is that we can let X act on itself by left and right translations, which are rational maps. Left translations commute with right translations, and the group we want to construct can be obtained as the group of birational maps from X to X that is generated by the left translations, or, equivalently, the group of birational maps from X to X that commute with the right translations.

Definition 2.2. – We let $\mathscr{R}$ be the set of birational maps from X to itself, that is, the set of equivalence classes of (U, f, V), where U and V are open and dense in X and $f \colon U \to V$ is an isomorphism, where (U, f, V) is equivalent to (U', f', V') if and only if f and f' are equal on $U \cap U'$ (note that X is separated, this is needed for transitivity of the relation). For each element g of $\mathscr{R}$ there is a maximal dense open subset $\mathrm{Dom}(g)$ of X on which it is a morphism.

Remark 2.3. – The elements of $\mathscr{R}$ can be composed, they have inverses, and so $\mathscr{R}$ is a group. For (U, f, V) as above, let g be $f^{-1} \colon V \to U$, then f and g induce inverse morphisms between $f^{-1}\,\mathrm{Dom}(g)$ and $g^{-1}\,\mathrm{Dom}(f)$, and therefore $(f^{-1}\,\mathrm{Dom}(g), f, g^{-1}\,\mathrm{Dom}(f))$ is a maximal representative of the equivalence class of (U, f, V) (see the proof of Lemma 3.6 for details).

Lemma 2.4. – *For a in X, let $U_a := (a, \mathrm{id}_X)^{-1} U_{1,2}$ and $V_a := (a, \mathrm{id}_X)^{-1} U_{1,3}$. Then U_a and V_a are open and dense in X, and $f_{1,2} \circ (a, \mathrm{id}_X) \colon U_a \to X$, $x \mapsto ax$, and $f_{1,3} \circ (a, \mathrm{id}_X) \colon V_a \to X$ induce inverse morphisms between U_a and V_a.*