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UNIPOTENT GROUPS OVER A DISCRETE VALUATION RING (AFTER DOLGACHEV-WEISFEILER)

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Abstract. – The aim of these notes is to understand the important work [**22**] of Dolgachev-Weisfeiler on unipotent group schemes over a discrete valuation ring. Among the results presented in these notes, we show, following [**22**], the existence of good generators of the affine ring of such a group scheme.

Résumé **(Groupes unipotents sur un anneau de valuation discret (d'après Dolgachev-Weisfeiler))**
Le but de ces notes est de comprendre le travail important [**22**] de Dolgachev-Weisfeiler sur les schémas en groupes unipotents sur un anneau de valuation discrète. Parmi les résultats présentés dans ces notes, nous démontrons, suivant [**22**], l'existence de bons générateurs de l'anneau affine d'un tel schéma en groupes.

Introduction

The theory of unipotent algebraic groups, and in particular of commutative unipotent algebraic groups, over a field represents a very beautiful theory (see for example [**5**] [**13**] [**16**] [**18**]), which plays also an important role in the study of algebraic groups over a field. Hence we can also expect that over a general base scheme, a study of unipotent group schemes can give applications in the study of families of algebraic groups. On the other hand, families of unipotent groups arise also naturally in practice, and lead to interesting questions about affine schemes.

In Exposé XVII of [**7**], a general theory of unipotent groups over a field is given. Besides of this, one can still find in Exposé XXVI of [**8**] some studies of families of unipotent groups, but only in a very special case, namely, the unipotent radical of a parabolic subgroup of some reductive group. In [**22**], Dolgachev and Weisfeiler proposed a theory of unipotent groups in a more general setting. More precisely, they considered affine unipotent group schemes G flat over an affine integral scheme $S = \mathrm{Spec}(R)$ such that the generic fiber is isomorphic to an affine space as a scheme,

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and obtained many interesting properties about such group schemes. For example, if R is a discrete valuation ring, the authors were able to find a family of good generators of the affine ring $R[G]$ of G/S and determined all the relations between these generators. When R is of equal characteristic $p > 0$ and if $G_K \simeq \mathbb{G}_{a,K}^n$ as a group, they proved that the group scheme G/S is a so-called p-polynomial S-scheme. With this result, assuming moreover that G/S has smooth connected fibers, together with some computations with the p-polynomials, the authors proved that the group scheme G/S is isomorphic to $\mathbb{G}_{a,S}^n$ possibly after an extension of discrete valuation rings of R. In addition to these, one finds in **[22]** also many results concerning deformations and cohomology of such group schemes.

The present report is then an attempt to understand the paper **[22]** of Dolgachev-Weisfeiler. Since the majority of the results in **[22]** are based on the assumption that the base scheme S is the spectrum of a discrete valuation ring, we will work mainly over such a base here. These notes contain *no* original result, and every statement here is contained in **[22]** (or **[5]**, **[14]**), though in some places, the treatments given here are slightly different from the original ones. But of course, I am responsible for any error in this work.

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1. Notations and reviews

1.1. Notations and conventions

1.1.1. – Unless mentioned explicitly, in this report, the letter R denotes a discrete valuation ring, K its fraction field and k its residue field. Moreover, we denote by S the spectrum of R.

1.1.2. – For $m, n \in \mathbb{Z}$ two integers such that $m \leq n$, by abuse of notation, we will denote by $[m, n]$ the set $\{m, m+1, \ldots, n\} \subset \mathbb{Z}$.

1.1.3. – Let $i \in [1, n]$, and for $r \in \mathbb{Z}_{\geq 0}$, we define

$$m(i, r) = (0, \ldots, 0, r, 0, \ldots, 0) \in \mathbb{Z}_{\geq 0}^n,$$

where the integer r is located in the i-th component.

1.1.4. – For G an affine group scheme over an affine scheme $\mathrm{Spec}(A)$, we denote by $A[G]$ the affine ring of G, and by

$$\mu : A[G] \to A[G] \otimes_A A[G]$$

its comultiplication. Moreover, we denote by $\eta : A[G] \to A[G] \otimes_A A[G]$ the morphism given by

$$x \mapsto \mu(x) - x \otimes 1 - 1 \otimes x.$$

1.2. Unipotent groups over a field: definitions and examples

1.2.1. – Let k be an *algebraically closed* field, and G a group scheme over k. Recall that G/k is *unipotent* if the following two equivalent conditions hold:

- G/k has a central composition series of the form

  $$0 = H_0 \subset H_1 \subset \cdots H_{r-1} \subset H_r = G,$$

  whose successive quotients H_i/H_{i-1} for $i = 1, 2, \ldots, r$ are isomorphic to the algebraic subgroups of $\mathbb{G}_{a,k}$ ([**7**] Exposé XVII, Définition 1.1).
- G is affine, and there exist generators $t_1, \ldots, t_n$ of $k[G]$ as a k-algebra such that the comultiplication μ verifies

  $$\mu(t_i) = t_i \otimes 1 + 1 \otimes t_i + \sum_j a_{ij} \otimes b_{ij} \quad (\forall i = 1, \ldots, n),$$

  with $a_{ij}, b_{ij} \in k[t_1, \ldots, t_{i-1}]$ ([**17**] Chap. VII § 1.6, Remarque 2).

1.2.2. – More generally, let k be an arbitrary field, with $\bar{k}$ an algebraic closure of k. A group scheme G/k is called *unipotent* if its base change $G_{\bar{k}} := G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ is a unipotent group over $\bar{k}$ in the sense of § 1.2.1. When the group scheme G/k is smooth and connected, this last condition is also equivalent to the following one ([**7**] Exposé XVII, Proposition 4.1.1):

- G has a composition series whose successive quotients are forms of $\mathbb{G}_{a,k}$.

In particular, once the base field k is *perfect*, since there is no non-trivial form of $\mathbb{G}_{a,k}$ over k ([**7**] Exposé XVII, Lemme 2.3 bis), G has a composition series whose successive quotients are isomorphic to $\mathbb{G}_{a,k}$. As a result, its underlying scheme is isomorphic to $\mathbb{A}^n_k$ with $n = \dim(G)$. More generally, without the perfectness condition on the field k, we have the following result due to Lazard.

***Proposition 1.1* (Lazard [5] IV § 4 n° 4, Théorème 4.1).** – *Let G/k be an affine k-group scheme. The following three statements are equivalent:*

- *There is an isomorphism of k*-schemes *$G \simeq \mathbb{A}^n_k$ with $n = \dim(G)$;*
- *G has a composition series with successive quotients isomorphic to $\mathbb{G}_{a,k}$;*
- *G is reduced and solvable. Moreover, there exist an integer N and a dominant morphism of k-schemes $\mathbb{A}^N_k \to G$.*

Definition 1.2 **([19] Chap. IV Definition 4.1.2, [4] Appendix B § B.1)**
*A unipotent group G/k is call a k-*split *(or* split*) unipotent group, if G/k verifies one of the three equivalent conditions in Proposition 1.1.*

Hence, for G/k a *split* unipotent group over k of dimension n, one can find generators $x_1, \ldots, x_n$ of $k[G]$ such that its comultiplication $\mu : k[G] \to k[G] \otimes k[G]$ satisfies

$$\mu(x_i) = x_i \otimes 1 + 1 \otimes x_i + \sum_j a_{ij} \otimes b_{ij}$$

with $a_{ij}, b_{ij} \in k[x_1, \ldots, x_{i-1}]$. In this report, we are mainly interested in such unipotent groups and their flat affine models over a discrete valuation ring.

1.2.3. *Some examples of unipotent groups.* – Let G be a smooth connected unipotent group over a field k of characteristic $p > 0$.

1. If $\dim(G) = 1$, then G is a form of the additive group $\mathbb{G}_{a,k}$. Hence if k is perfect, $G \simeq \mathbb{G}_{a,k}$ (**[7]** Exposé XVII, Lemme 2.3 bis). But over an imperfect field, there exists non-trivial form of $\mathbb{G}_{a,k}$. For example, let $a \in k - k^p$, and consider the closed subgroup scheme G of $\mathbb{G}_k^2 = \mathrm{Spec}(k[x,y])$ defined by the equation
$$x + x^p + ay^p = 0.$$
Then G is a non-trivial form of $\mathbb{G}_{a,k}$, which can be trivialized by the inseparable extension $k \subset k(a^{1/p})$ (i.e.,, the base change $G \otimes_k k(a^{1/p})$ is isomorphic to $\mathbb{G}_{a,k(a^{1/p})}$ as a group scheme over $k(a^{1/p})$).
2. For $r \in \mathbb{Z}_{\geq 1}$, let
$$\Phi_r(X,Y) = \frac{1}{p} \sum_{i=1}^{p^r-1} \binom{p^r}{i} X^i Y^{p^r-i} \in \mathbb{Z}[X,Y].$$
We consider the k-algebra of polynomials in two variables $k[x,y]$, and define the map $\mu : k[x,y] \to k[x,y] \otimes k[x,y]$ by:
$$\mu(x) = x \otimes 1 + 1 \otimes x, \quad \mu(y) = y \otimes 1 + 1 \otimes y + \sum_{r \geq 1} a_r \Phi_r(x \otimes 1, 1 \otimes x),$$
with $a_r \in k$ such that $a_r = 0$ for almost all r. We verify that this gives a structure of Hopf algebra on $k[x,y]$, and the group scheme obtained in this way is an extension of $\mathrm{Spec}(k[x]) = \mathbb{G}_{a,k}$ by $\mathrm{Spec}(k[x,y]/(x)) = \mathbb{G}_{a,k}$, which is also commutative. Conversely, any two dimensional *commutative split* unipotent group is given in this way. Indeed, such a split unipotent group can be obtained as an extension of $\mathbb{G}_{a,k}$ by itself over k (Definition 1.2). On the other hand, the group of extensions $\mathrm{Ext}_k^1(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ in the category of abelian fppf-sheaves over k can be described by the symmetric Hochschild cohomology group $\mathrm{H}_s^2(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ (**[5]** II § 3 2.4 Proposition). The latter cohomology group has a natural structure of k-vector space, and by **[5]** II § 3 4.6 Théorème, it is generated over k by the classes of the polynomials $\Phi_r(X,Y)$ $(r = 1, 2 \cdots)$. Using the identification between $\mathrm{Ext}_k^1(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ and $\mathrm{H}_s^2(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ (**[5]** II § 3 2.3), this last assertion