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SMOOTH MODELS ASSOCIATED TO CONCAVE FUNCTIONS IN BRUHAT-TITS THEORY

by

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Abstract. – In Bruhat-Tits theory, a systematic construction of bounded subgroups is given using concave functions f. When $f(0) = 0$, the corresponding bounded subgroup can be enriched to a group scheme over the ring of integers in the base local field. We give a new treatment of the existence of these group schemes while removing the hypothesis $f(0) = 0$, so that the result can be applied to many bounded subgroups used in representation theory, including the Moy-Prasad groups.

Résumé **(Modèles lisses associés aux fonctions concaves en théorie de Bruhat-Tits)**
La théorie de Bruhat-Tits donne lieu à une construction systématique de sous-groupes bornés au moyen de fonctions concaves f. Dans le cas où $f(0) = 0$, le groupe borné correspondant peut être enrichi d'une structure de schéma en groupes sur l'anneau d'entiers du corps local de base. Nous donnons une nouvelle construction de ces schémas en groupes en levant l'hypothèse $f(0) = 0$, de sorte que le résultat est applicable à une famille étendue de sous-groupes bornés utilisés en théorie des représentations, notamment les groupes de Moy-Prasad.

0. Introduction

Let $\mathscr{O}$ be a Henselian discrete valuation ring with perfect residue field, and let G be a quasi-split connected reductive group over k, the quotient field of $\mathscr{O}$. Let S be a maximal k-split torus of G, $\Phi = \Phi(G, S)$ the corresponding root system. For simplicity, in this introduction we assume that $\Phi(G, S)$ is reduced.

Fix a point x on the apartment $A(G, S)$ attached to S. According to Bruhat-Tits theory, x determines a filtration $\{U_a(k)_{x,r}\}_{r\in\mathbb{R}}$ for each root subgroup U_a of G. These groups can be used to form very well-behaved bounded open subgroups of the totally disconnected group $G(k)$, which is a key objective of Bruhat-Tits theory. To proceed, let $f : \Phi \cup \{0\} \to \mathbb{R}$ be a concave function (see 8.1 for the definition) with $f(0) = 0$, and let $G(k)_{x,f}$ be the subgroup generated by $U_a(k)_{x,f(a)}$ for all $a \in \Phi \cup \{0\}$ with the convention that $U_0 = Z_G(S)$ and $U_0(k)_{x,0}$ is the Iwahori subgroup of $U_0(k)$

(see 1.4.2). Then $G(k)_{x,f}$ is a bounded open subgroup satisfying many useful group-theoretic properties (see [**2**, 6.4]). Bruhat-Tits also proved the following fundamental algebro-geometric result:

Theorem 0.1. – *There is a canonical affine smooth group scheme $\underline{G}_{x,f}$ over $\mathscr{O}$, with generic fiber G, such that*

(i) $\underline{G}_{x,f}(\mathscr{O}) = G(k)_{x,f}$;
(ii) *for each $a \in \Phi \cup \{0\}$, the schematic closure $\underline{U}_a$ of U_a in $\underline{G}_{x,f}$ is smooth;*
(iii) *for any system Φ^+ of positive roots in Φ, the multiplication morphism*

$$\Big(\prod_{a\in\Phi^-} \underline{U}_a\Big) \times \underline{U}_0 \times \Big(\prod_{a\in\Phi^+} \underline{U}_a\Big) \to \underline{G}_{x,f},$$

is an open immersion, where the two products $\prod_{a\in\Phi^\pm}$ can be taken in any order.

For example, by taking $f(a) = 0$ for all $a \in \Phi \cup \{0\}$, we get the group scheme canonically associated to the (connected) parahoric subgroup $G(k)_{x,0}$ of $G(k)$.

However, the assumption $f(0) = 0$ is quite restrictive from certain point of view, e.g. it hinders one from discussing natural results concerning congruence subgroups in this context (see 8.3.2). Two possible reasons for this restriction are as follows. Firstly, Bruhat-Tits did not put a filtration on $U_0(k)$ in [**3**], although preliminary consideration was made in [**2**, 6.4]. The Iwahori subgroup $U_0(k)_{x,0}$ should be regarded as the level zero filtration group of $U_0(k)$. We can only consider f with $f(0) = 0$ without defining $U_0(k)_{x,r}$ in general.

Secondly, the assumption $f(0) = 0$ also seems to come from an intrinsic limitation of the framework of Bruhat-Tits to construct the associated group schemes from their theory of schematic root datum (see [**3**, 3.3.1 (DRS 0)]).

Moy-Prasad and Schneider-Stuhler have defined a filtration on $U_0(k)$, making it possible to consider $G(k)_{x,f}$ for arbitrary concave f. In particular, the constant function f with $f(a) = r$ for all $a \in \Phi \cup \{0\}$ gives the Moy-Prasad group $G(k)_{x,r}$, which has been extremely useful in the representation theory of p-adic groups.

0.1. – The purpose of this paper is three-fold. First, we extend Theorem 0.1 to the case of an arbitrary concave function f (removing the hypothesis $f(0) = 0$). In particular, we construct canonical smooth models associated to Moy-Prasad groups and Schneider-Stuhler groups.

Our result provides a link between the Moy-Prasad filtrations on a p-adic group and those on its Lie algebra. In addition, it should facilitate employing machinery of algebraic geometry into representation theory.

0.2. – The second purpose of this paper is to offer a treatment of Theorem 0.1 with minimal dependence on [**3**]. For most applications of Bruhat-Tits theory (e.g. to number theory and representation theory), the current paper may replace a good part of [**3**]. Our approach is completely different from that of [**3**], and is much simpler. It is also different from the approach using the Artin-Weil theorem, hinted in [**3**, 3.1.7] and carried out in [**11**]. The main idea is to use the theory of dilatation and group smoothening systematically [**1**].

To be more precise, we now recall the organization of the monumental work of Bruhat and Tits [**2**], [**3**]. Chapter 1 ([**2**]) is written in the language of abstract group theory. There, the notion of a valuation ([**2**, 6.2]) of a group $\mathscr{G}$ with root datum ([**2**, 6.1]) is defined and explored. Whenever we have such a valuation, there is an associated affine building, and actually every point on the building gives rise to a valuation of root datum. We can then derive a lot of geometric/algebraic structures: the apartments, the polysimplicial structure, the double Tits system, the parahoric subgroups, the bounded subgroups associated to concave functions, and so on.

In Chapter 2, it is shown that the theory of Chapter 1 applies to a connected reductive group G over k. That is, there are canonical valuations of root datum on $\mathscr{G} = G(k)$. The theory of étale descent ([**3**, §5]) reduces the question to the case of a quasi-split G. In that case, a valuation of root datum can be written down explicitly using a Chevalley-Steinberg system. This is done in [**3**, 4.1, 4.2]. One can also use [**14**] and [**11**] for alternative treatments. Once this is taken care of, we have all the structures introduced in Chapter 1.

But the bulk of Chapter 2 is to show that now there are even more structures of algebro-geometric nature: the abstract bounded subgroups introduced in Chapter 1 underly canonical group schemes. One of the main result of Chapter 2 is Theorem 0.1 (with $f(0) = 0$). It is this part that we are able to replace and generalize.

We remark that the results of Bruhat-Tits are more general in that they allows fields with non-discrete valuations. Also, their theory of schematic root datum works over fairly general integral domains (hence gives a nice construction of Chevalley schemes over $\mathbb{Z}$, for example). Our goal is only to give a proof of Theorem 0.1 over a Henselian discrete valuation ring.

0.3. – The third purpose of this paper is to offer a replacement of the Moy-Prasad filtration for a torus. The original Moy-Prasad filtration has some anomalies (see 4.5) which are undesirable from the viewpoint of algebraic geometry. Also, the so-called Moy-Prasad isomorphism is valid only under certain tameness condition (which is satisfied in [**12**]). This isomorphism is needed for the proofs of [**13**]. Recently, DeBacker [**5**] has proved most results in [**13**] without using the isomorphism. However, the isomorphism is useful for many other purposes.

The minimal congruent filtration defined in this paper resolves all these problems. However, to make our theory more applicable, we axiomize a class of filtrations, called the schematic admissible filtrations, which include both the original Moy-Prasad filtration and the minimal congruent filtration. We prove the main theorem for all schematic admissible filtrations.

0.4. – This paper is organized as follows. Section 1 summarizes some notations. In Section 2, we collect facts about smooth models and smoothening. In Section 3 we give a short proof of the existence of smooth models associated to parahoric subgroups, which will serve as the starting point of our inductive construction of the group scheme $\underline{G}_{x,f}$.

Sections 4, 5, 6 consider filtrations on $U_0(k)$, where $U_0 = T$ is a torus. The matter is quite delicate and technical. They can be skipped in first reading, in particular by those who only want to see our proof of Theorem 0.1 with $f(0) = 0$.

Section 7 prepares some technical results for the proof of Theorem 0.1 in Section 8. The idea is quite simple: it is intuitive that $\underline{G}_{x,f}(\mathscr{O})$ contains the first congruence subgroup of $\underline{G}_{x,g}(\mathscr{O})$ when $g \leq f \leq g+1$. Therefore $\underline{G}_{x,f}$ should be constructed as a dilatation on $\underline{G}_{x,g}$ (see 2.3). This would definitely satisfy (i) but we also need to arrange matters to faciliate the proof of (ii) and (iii) in this induction scheme.

Section 9 contains various additional results, including some remark on Schneider-Stuhler theory.

In Section 10, we examine the compact groups used in the author's earlier work on construction of tame supercuspidal representations [**23**]. It turns out that these groups are not necessarily of the form $G(k)_{x,f}$, but they also admits canonical smooth models. Moreover, the inducing representations on these compact groups have a very simple and special form, indicating that they might be studied by algebraic geometry.

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1. Notations

1.1. – Let k be a field with a Henselian discrete valuation $\mathrm{ord} : k^\times \to \mathbb{Z}$. Let $\mathscr{O}$ be the ring of integers in k. We assume that the residue field κ of $\mathscr{O}$ is perfect. We denote by $\tilde{\mathscr{O}}$ the strict henselization of $\mathscr{O}$, and $\tilde{k}$ the field of fractions of $\tilde{\mathscr{O}}$. Finally, π is a fixed prime element of $\mathscr{O}$. The valuation is normalized so that $\mathrm{ord}(\pi) = 1$.