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HEEGAARD FLOER HOMOLOGIES

by

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Abstract. – These lecture notes are an introduction to Heegaard Floer homology, a collection of tools in low-dimensional topology introduced by Ozsváth-Szabó and others. We focus on Juhasz's sutured Heegaard Floer homology as a common framework for many of the Heegaard Floer invariants. In the first three lectures we sketch the definition of sutured Heegaard Floer homology and proofs of some of its main properties. In the fourth lecture we discuss an application of Floer theory to knot surgery, originally proved by Kronheimer-Mrowka-Ozsváth-Szabó, and a connection discovered by Ozsváth-Szabó between Heegaard Floer homology and Khovanov homology.

***Résumé* (Homologies de Heegaard Floer).** – Ces notes sont une introduction à l'homologie de Heegaard Floer, une collection d'outils en topologie de basse dimension développée par Ozsváth, Szabó et d'autres auteurs. Nous nous concentrons sur l'homologie de Heeagaard Floer suturée de Juhasz qui fournit un cadre commun pour étudier beaucoup des invariants de Heegaard Floer. Dans les trois premiers cours, nous donnons les grandes lignes de la définition de l'homologie de Heegaard Floer suturée et des démonstrations de certaines de ses propriétés principales. Dans le quatrième cours, nous présentons une application dûe à Kronheimer, Mrowka, Ozsváth et Szabó de l'homologie de Floer à la chirurgie sur les nœuds, et une relation découverte par Ozsváth et Szabó entre l'homologie de Heegaard Floer et l'homologie de Khovanov.

1. Introduction and overview

1.1. A brief overview. – Heegaard Floer homology is a family of invariants of objects in low-dimensional topology. The first of these invariants were introduced by Ozsváth-Szabó: invariants of closed 3-manifolds and smooth 4-dimensional cobordisms [**58, 63**] (see also [**27**]). Later, Ozsváth-Szabó and, independently, Rasmussen introduced invariants of knots in 3-manifolds [**56, 73**]. There are also several other invariants, including invariants of contact structures [**59**], more invariants of knots and

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3-manifolds [**62, 54**], and invariants of Legendrian and transverse knots [**70, 16, 44**]. The subject has had many applications; I will not even try to list them here, though we will see a few in the lectures.

In the first three of these lectures, we will focus on a generalization of one variant of these invariants: an invariant of sutured 3-manifolds, due to Juhász, called *sutured Floer homology* [**23**]. The main goal will be to relate these invariants to ideas in more classical 3-manifold topology. In particular, we will sketch a proof that sutured Floer homology detects the genus of a knot. The proof, which is due to Juhász [**24**] extending earlier results of Ozsváth-Szabó [**55**], uses Gabai's theory of sutured manifolds and sutured hierarchies, which we will review in the first lecture.

In the fourth lecture, we go in a different direction: we will talk about the surgery exact sequence in Heegaard Floer homology. The goal is to sketch a (much studied) relationship between Heegaard Floer homology and Khovanov homology: a spectral sequence due to Ozsváth-Szabó [**62**].

1.2. A more precise overview. – Heegaard Floer homology assigns to each closed, oriented, connected 3-manifold Y an abelian group $\widehat{HF}(Y)$, and $\mathbb{Z}[U]$-modules $HF^+(Y)$, $HF^-(Y)$ and $HF^\infty(Y)$. These are the homologies of chain complexes $\widehat{CF}(Y)$, $CF^+(Y)$, $CF^-(Y)$ and $CF^\infty(Y)$. These chain complexes are related by short exact sequences

$$0 \longrightarrow CF^-(Y) \xrightarrow{\cdot U} CF^\infty(Y) \longrightarrow CF^+(Y) \longrightarrow 0$$
$$0 \longrightarrow CF^-(Y) \xrightarrow{\cdot U} CF^-(Y) \longrightarrow \widehat{CF}(Y) \longrightarrow 0$$
$$0 \longrightarrow \widehat{CF}(Y) \longrightarrow CF^+(Y) \xrightarrow{\cdot U} CF^+(Y) \longrightarrow 0$$

which, of course, induce long exact sequences in homology. In particular, either of $CF^+(Y)$ or $CF^-(Y)$ determines $\widehat{CF}(Y)$. (The complexes $CF^+(Y)$ and $CF^-(Y)$ also have equivalent information, though this does not quite follow from what we have said so far.) These invariants are defined in [**58**]. (Some students report finding it helpful to read[**38**] in conjunction with [**58**].) It is now known, by work of Hutchings [**19**], Hutchings-Taubes [**20, 21**], Taubes[**76, 77, 78, 79, 80**], and Kutluhan-Lee-Taubes[**33, 34, 35, 36, 37**] or Colin-Ghiggini-Honda[**4, 5, 3**], that these invariants correspond to different variants of Kronheimer-Mrowka's Seiberg-Witten Floer homology groups [**29**].

Roughly, smooth, compact, connected 4-dimensional cobordisms between connected 3-manifolds induce chain maps on $\widehat{CF}$, $CF^\pm$ and CF^∞, and composition of cobordisms corresponds to composition of maps. From the maps on $CF^\pm$ and the exact sequences above, one can recover the Seiberg-Witten invariant, or at least something very much like it [**63**]. Note, in particular, that $\widehat{CF}$ does not have enough information to recover the Seiberg-Witten invariant.

There is an extension of the Heegaard Floer homology groups to nullhomologous knots in 3-manifolds, called *knot Floer homology* **[56, 73]**. Given a knot K in a 3-manifold Y there is an induced filtration of $\widehat{CF}(Y)$, $CF^+(Y)$, and so on. In particular, we can define the *knot Floer homology groups* $\widehat{HFK}(Y,K)$, the homology of the associated graded complex to the filtration on $\widehat{CF}(Y)$. (So, there is a spectral sequence from $\widehat{HFK}(Y,K)$ to $\widehat{HF}(Y)$.)

The gradings in the subject are quite subtle. The chain complexes $\widehat{CF}(Y)$, $CF^+(Y)$, ..., decompose as direct sums according to spin^c-*structures* on Y, i.e.,

$$\widehat{CF}(Y) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c(Y)} \widehat{CF}(Y,\mathfrak{s}).$$

(We will discuss spin^c structures more in Section 3.4.1.) Each of the $\widehat{CF}(Y,\mathfrak{s})$ is relatively graded by some $\mathbb{Z}/n\mathbb{Z}$ (where n is the divisibility of $c_1(\mathfrak{s})$). In particular, if $c_1(\mathfrak{s}) = 0$ (i.e., $\mathfrak{s}$ is *torsion*) then $\widehat{CF}(Y,\mathfrak{s})$ has a relative $\mathbb{Z}$ grading. Similarly, $\widehat{HFK}(Y,K)$ decomposes as a direct sum of groups, one per relative spin^c structure on (Y,K).

In the special case that $Y = S^3$, there is a canonical identification $\mathrm{spin}^c(S^3,K)\cong\mathbb{Z}$, and each $\widehat{HFK}(Y,K,\mathfrak{s})$ in fact has an absolute $\mathbb{Z}$-grading. That is, $\widehat{HFK}(Y,K)$ is a bigraded abelian group. We will write $\widehat{HFK}(S^3,K) = \widehat{HFK}(K) = \bigoplus_{i,j}\widehat{HFK}_i(K,j)$, where j stands for the spin^c grading. The grading j is also called the *Alexander grading*, because

$$\sum_{i,j}(-1)^i t^j \operatorname{rank}\widehat{HFK}_i(K,j) = \Delta_K(t),$$

the (Conway normalized) Alexander polynomial of K.

The breadth of the Alexander polynomial $\Delta_K(t)$, or equivalently the degree of the symmetrized Alexander polynomial, gives a lower bound on the genus $g(K)$ of K (i.e., the minimal genus of any Seifert surface for K). One of the main goals of these lectures will be to sketch a proof of the following refinement:

Theorem 1.1. – **[55]** *Given a knot K in S^3,*

$$g(K) = \max\{j \mid \big(\bigoplus_i \widehat{HFK}_i(K,j)\big)\neq 0\}.$$

Rather than giving the original proof of Theorem 1.1, we will give a proof using an extension of $\widehat{HF}$ and $\widehat{HFK}$, due to Juhász, called *sutured Floer homology.* Sutured manifolds were introduced by Gabai in his work on foliations, fibrations, the Thurston norm, and knot genus **[7, 8, 9, 10]**; we will review some aspects of this theory in the first lecture. Sutured Floer homology associates to each sutured manifold (Y,Γ) satisfying certain conditions (called being *balanced*) a chain complex $SFC(Y,\Gamma)$ whose homology $SFH(Y,\Gamma)$ is an invariant of the sutured manifold. These chain complexes behave in a particular way under Gabai's *surface decompositions*, leading to a proof of Theorem 1.1.

In the last lecture, we turn to a different topic: the behavior of Heegaard Floer homology under knot surgery. The goal is to relate these lectures to the lecture series on Khovanov homology. In particular, we will sketch the origins of Ozsváth-Szabó's spectral sequence $\widetilde{Kh}(m(K)) \Rightarrow \widehat{HF}(\Sigma(K))$ from the (reduced) Khovanov homology of the mirror of K to the Heegaard Floer homology of the branched double cover of K [**62**].

1.3. References for further reading. – There are a number of survey articles on Heegaard Floer homology. Three by Ozsváth-Szabó [**60, 64, 65**] give nice introductions to the Heegaard Floer invariants of 3- and 4-manifolds and knots. Juhász's recent survey [**26**] contains an introduction to sutured Floer homology, which is the main subject of these lectures. There are also some more focused surveys of other recent developments [**45, 41**].

Sutured Floer homology, as we will discuss it, is developed in a pair of papers by Juhász [**23, 24**]. For a somewhat different approach to relating sutured manifolds and Floer theory, see the work of Ni (starting perhaps with [**51**]).

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2. Sutured manifolds, foliations and sutured hierarchies

2.1. The Thurston norm and foliations

Definition 2.1. – *Given a knot $K \subset S^3$, the* genus of K *is the minimal genus of any Seifert surface for K (i.e., of any embedded surface $F \subset S^3$ with $\partial F = K$).*

Thurston found a useful generalization of this notion to arbitrary 3-manifolds and, more generally, to link complements in arbitrary 3-manifolds:

Definition 2.2. – *For a 3-manifold Y with boundary ∂Y a (possibly empty) disjoint union of tori, the* Thurston norm

$$x\colon H_2(Y, \partial Y) \to \mathbb{Z}$$

is defined as follows. Given a compact, oriented surface F (not necessarily connected, possibly with boundary) define the complexity of F *to be*

$$x(F) = \sum_{\chi(F_i)\leq 0} |\chi(F_i)|,$$

where the sum is over the connected components F_i of F.

Given an element $h \in H_2(Y, \partial Y)$ *and a surface* $F \subset Y$ *with* $\partial F \subset \partial Y$ *we say that* F represents h *if the inclusion map sends the fundamental class of* F *in* $H_2(F, \partial F)$ *to* h*. Define*

$$x(h) = \min\{x(F) \mid F \text{ an embedded surface representing } h\}.$$

For this definition to make sense, we need to know the surface F exists:

Lemma 2.3. – *Any element* $h \in H_2(Y, \partial Y)$ *is represented by some embedded surface* F*.*

Idea of Proof. – The class h is Poincaré dual to a class in $H^1(Y)$, which in turn is represented by a map $f_h\colon Y \to K(\mathbb{Z}, 1) = S^1$. The preimage of a regular value of f_h represents h. See for instance [**81**, Lemma 1] for more details. □

Proposition 2.4. – *If* $(Y, \partial Y)$ *has no essential spheres (*Y *is* irreducible*) or disks (*∂Y *is* incompressible*) then* x *defines a pseudo-norm on* $H_2(Y, \partial Y)$ *(i.e., a norm except for the non-degeneracy axiom). If moreover* Y *has no essential annuli or tori (*Y *is* atoroidal*) then* x *defines a norm on* $H_2(Y, \partial Y)$*, and induces a norm on* $H_2(Y, \partial Y; \mathbb{Q})$*.*

Idea of Proof. – The main points to check are that:

1. $x(n \cdot h) = n \cdot x(h)$ for $n \in \mathbb{N}$.
2. $x(h + k) \leq x(h) + x(k)$.

For the first point, a little argument shows that a surface representing $n \cdot h$ (with h indivisible) necessarily has n connected components, each representing h. The second is a little more complicated. Roughly, one takes surfaces representing h and k and does surgery on their circles and arcs of intersection to get a new surface representing $h + k$ without changing the Euler characteristic. (More precisely, one first has to eliminate intersections which are inessential on both surfaces, as doing surgery along them would create disjoint S^2 or $\mathbb{D}^2$ components.) See [**81**, Theorem 1] for details. □

Example 2.5. – If $Y = S^3 \setminus \mathrm{nbd}(K)$ is the exterior of a knot then $H_2(Y, \partial Y) \cong \mathbb{Z}$ and surfaces representing a generator for $H_2(Y, \partial Y)$ are Seifert surfaces for K. The Thurston norm of a generator is given by $2g(K) - 1$ (if K is not the unknot).

Example 2.6. – Consider $Y = S^1 \times \Sigma_g$, for any $g > 0$. Fix a collection of curves γ_i, $i = 1, \dots, 2g$, in Σ giving a basis for $H_1(\Sigma)$. Then $H_2(Y) \cong \mathbb{Z}^{2g+1}$, with basis (the homology classes represented by) $S^1 \times \gamma_i$, $i = 1, \dots, 2g$, and $\{pt\} \times \Sigma$. We have $x([S^1 \times \gamma_i]) = 0$, from which it follows (why?) that x is determined by $x([\{pt\} \times \Sigma])$. One can show using elementary algebraic topology that $x([\{pt\} \times \Sigma]) = 2g - 2$; see Exercise 1.

Remark 2.7. – A norm is determined by its unit ball. The Thurston norm ball turns out to be a polytope defined by inequalities with integer coefficients [**81**, Theorem 2].

A priori, the Thurston norm looks impossible to compute in general. Remarkably, however, it can be understood. The two key ingredients are foliations, which we discuss now, and a decomposition technique, due to Gabai, which we discuss next.